1.6 Conditional Probability

Sometimes, the probability of an event is affected by events that have already occurred.

Introductory Example: I have a jar of 10 marbles: 4 red and 6 green. Suppose I pick 2 marbles from the jar one at a time without replacement and observe the color of each marble.

- What is the probability that the first marble picked is green?
- What is the probability that the second marble picked is green given that the first one was red?
- What is the probability that the second marble picked is green given that the first one was green?

The second two probabilities above are called **conditional probabilities**.

$P(B|A)$ stands for the probability of the event $B$ given that the event $A$ has already occurred. It is the *conditional probability of $B$ given $A$*.

**Conditional Probability of an Event**

If $A$ and $B$ are events in an experiment and $P(A) \neq 0$, then the conditional probability that the event $B$ will occur given that $A$ has already occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Example: A pair of fair dice is rolled. What is the probability that the sum is more than 9, given that a 5 is rolled?

Example: What is the probability that a 5 is rolled given that the sum is more than 9?
Example: (modified from *Finite Mathematics* by Long/Graening) In a survey, 100 adults were asked their political party and whether they live in an urban, suburban, or rural area. The results are given in the table:

<table>
<thead>
<tr>
<th></th>
<th>Urban</th>
<th>Suburban</th>
<th>Rural</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Democrat</td>
<td>20</td>
<td>18</td>
<td>7</td>
<td>45</td>
</tr>
<tr>
<td>Republican</td>
<td>17</td>
<td>21</td>
<td>8</td>
<td>46</td>
</tr>
<tr>
<td>Independent</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>Total</td>
<td>42</td>
<td>40</td>
<td>18</td>
<td>100</td>
</tr>
</tbody>
</table>

Find the probability that:

- a person from this group is a Democrat given that they live in an urban area.

- a person who lives in a suburban area is a Republican.

- an Independent lives in a rural area.

We can now write probabilities on the branches of tree diagrams to illustrate conditional probability.

Example: Draw a tree diagram for the introductory example above involving marbles: I have a jar of 10 marbles: 4 red and 6 green. Suppose I pick 2 marbles in succession out of the jar at random without replacement and observe the color of each marble.

**Product Rule**

\[ P(A \cap B) = P(A) \cdot P(B|A) \]

Note that this is just a rearrangement of the formula for conditional probability.
Example: In a survey of 1000 eligible voters selected at random, it was found that 80 had a college degree. Additionally, it was found that 80% of those who had a college degree voted in the last presidential election, whereas 55% of the people who did not have a college degree voted in the last presidential election.

Assuming the poll is representative of all eligible voters, find the probability that an eligible voter selected at random

- Had a college degree and voted in the last presidential election.

- Did not have a college degree and did not vote in the last presidential election.

- Voted in the last presidential election.

Example: A city police department reported that, of all vehicles stolen, 64% were stolen by professionals, whereas 36% were stolen by amateurs. Of those vehicles stolen by professionals, 24% were recovered within 48 hrs, 16% were recovered after 48 hrs, and 60% were never recovered. Of those vehicles stolen by amateurs, 38% were recovered within 48 hrs, 58% were recovered after 48 hrs, and 4% were never recovered.

- Draw a tree diagram for this data.
• What is the probability that a vehicle was stolen by an amateur and will be recovered after 48 hrs?

• What is the probability that a vehicle stolen by a professional will be recovered within 48 hrs?

• What is the probability that a vehicle will never be recovered?

Independence
Two events $A$ and $B$ are called independent events if the outcome of one event does not affect the outcome of the other. In mathematical terms, if $A$ and $B$ are independent, then

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

To test for independence of events, use the following:

**Two events $A$ and $B$ are independent if and only if**

$$P(A \cap B) = P(A) \cdot P(B)$$

An example of independent events is tossing a coin twice. Whether the second toss is heads or tails is not affected by whether the first toss is heads or tails.

Example: Suppose $A$ and $B$ are independent events with $P(A^c) = 0.7$ and $P(B) = 0.6$. What are $P(A \cap B)$ and $P(A \cup B)$?
Do not confuse independence with two events being mutually exclusive.

Example: Recall the example from before:

<table>
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- Are being an Independent and living in an urban area independent events for this group of people?

- Are being a Democrat and living in a suburban area independent events for this group of people?

Example: Kristina, Carlos, and Ben live in 3 different cities. The probability that Kristina goes to the store on a given day is 0.23, the probability that Carlos goes to the store on a given day is 0.15, and the probability that Ben goes to the store on a given day is 0.03. What is the probability that:

(a) All 3 will go to the store on a given day?

(b) None of them will go to the store?

(c) Exactly one of them will go to the store?
1.7 Bayes’ Theorem

Probabilities are sometimes calculated after the outcomes of an experiment have been observed.

To find these kinds of probabilities, we still just use the conditional probability formula.

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

Example: Research determined that 42% of 12-yr-olds have never had a cavity, 24% of 13-yr-olds have never had a cavity, and 28% of 14-yr-olds have never had a cavity. A child is selected at random from a group of 24 junior high school students made up of 6 12-yr-olds, 8 13-yr-olds, and 10 14-yr-olds. If this child does not have a cavity, what is the probability that he/she is 14 years old?

Example: A poll was conducted among 255 men and 245 women in a certain area regarding their position on a national lottery to raise revenue for the government. The results are in the following table:

<table>
<thead>
<tr>
<th>Answer</th>
<th>Males, %</th>
<th>Females, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Favor</td>
<td>62</td>
<td>68</td>
</tr>
<tr>
<td>Do Not Favor</td>
<td>32</td>
<td>28</td>
</tr>
<tr>
<td>No Opinion</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Draw a tree diagram for this data.

- What is the probability that a male voter did not favor a national lottery?

- What is the probability that a voter who favored a national lottery was a female?
Example: In a large city, 64% of the people are right-handed and the remaining people are left-handed. Of those who are right-handed, 42% are over 6 ft tall and 55% are between 5 and 6 ft tall inclusive. Of those who are left-handed, 7% are shorter than 5 ft tall. It is also known that 18% of the people in this city are left-handed and over 6 ft tall. Draw a tree diagram for this information.

- What is the probability that a right-handed person is between 5 and 6 ft tall inclusive?

- What is the probability that a person is left-handed if they are shorter than 5 ft tall?

- What is the probability that a person is right-handed or over 6 ft tall?

- Are being left-handed and over 6-ft tall independent events?
Example: I have 2 bags of marbles. Bag A has 4 reds, 3 blues, and 6 greens. Bag B has 5 reds, 2 blues, and 3 greens. An experiment consists of first selecting a marble from Bag A. If the marble is red, it is transferred to Bag B and then a second marble is picked from Bag B. If the first marble is blue, it is set aside and a second marble is picked from Bag A. If the first marble is green, it is set aside and a second marble is picked from Bag B.

- Find the probability that the second marble was red given that the first marble is green.

- If it is known that the second marble is blue, find the probability that the first marble was green.

Trees for medical tests: