5.1 Introduction to Matrices

Reminder: A matrix with \( m \) rows and \( n \) columns has size \( m \times n \). (This is also sometimes referred to as the order of the matrix.) The entry in the \( i \)th row and \( j \)th column of a matrix \( A \) is denoted by \( a_{ij} \).

If the number of rows equals the number of columns, we call the matrix a _________________.

Consider the matrix: 
\[
A = \begin{bmatrix}
1 & 7 & 3 & -3 \\
9 & 1 & -2 & 11 \\
-4 & 6 & 2 & 8
\end{bmatrix}
\]

What is the size of \( A \)? ________ What is \( a_{23} \)? ________ What is \( a_{12} \)? ________

Two matrices are equal if they have the same size and all their corresponding entries are equal.

We can add or subtract two matrices only if they have the same size. If they are the same size, we add or subtract them by adding or subtracting all their corresponding entries.

If we multiply a matrix by a scalar (constant), then every entry in the matrix is multiplied by this scalar. Multiplying by a scalar does NOT change the size of the matrix.

Example: Consider the matrices 
\[
A = \begin{bmatrix}
3 & 1 \\
2 & 4 \\
-4 & 0
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
2 & 6 \\
-1 & 3 \\
5 & -2
\end{bmatrix}
\]

Compute \(-2A + 3B\).

If \( A \) is an \( m \times n \) matrix with entries \( a_{ij} \), the transpose of \( A \), denoted \( A^T \), is the \( n \times m \) matrix found by making the rows of \( A \) the columns of \( A^T \).

Example: Let 
\[
A = \begin{bmatrix}
1 & -4 \\
-2 & 8 \\
3 & 5
\end{bmatrix}
\]

What is \( A^T \)?

Solve for the variables in the following matrix equation.

\[
2 \begin{bmatrix}
a + 3 & 4 \\
2 & 6
\end{bmatrix} - \begin{bmatrix}
-1 & 6 \\
9 & 7b - 1
\end{bmatrix} = \begin{bmatrix}
3 & d \\
c & -8
\end{bmatrix}^T
\]
Matrices are often used to organize and work with data, not solely for solving systems of equations.

Example: The number of science and non-science majors enrolled in Math 131, 151, and 166 were counted during two semesters. Matrix $F$ below gives data for the fall semester and matrix $S$ gives data for the spring semester.

\[
F = \begin{pmatrix}
M_{131} & M_{151} & M_{166} \\
248 & 492 & 324 \\
124 & 224 & 312
\end{pmatrix}
\quad S = \begin{pmatrix}
M_{131} & M_{151} & M_{166} \\
210 & 298 & 124 \\
320 & 258 & 110
\end{pmatrix}
\]

Find a matrix that gives the total number of science majors and the total number of non-science majors enrolled in each of these classes over the 1-year period.

Find a matrix that gives the average number of science majors and the average number of non-science majors enrolled in each of these classes in a semester over this 1-year period.

5.2 Matrix Multiplication

Multiplying matrices is not as easy as adding and subtracting them.

If $A$ is an $m \times n$ matrix and $B$ is an $r \times s$ matrix, then for the matrix product $AB$ to make sense, we must have that $n = r$. In other words, the number of columns in $A$ MUST equal the number of rows in $B$.

The resulting matrix, $AB$, is an $m \times s$ matrix.
To multiply $AB$, we move across the rows of $A$ as we move down the columns of $B$. We multiply the corresponding entries and add them up.

The entry in row 1, column 1 of $AB$ is found by using row 1 of $A$ and column 1 of $B$.
The entry in row 1, column 2 of $AB$ is found by using row 1 of $A$ and column 2 of $B$.
The entry in row 2, column 1 of $AB$ is found by using row 2 of $A$ and column 1 of $B$.
In general, if $C = AB$, then $c_{ij}$ [row $i$, column $j$] is found by using row $i$ of $A$ and column $j$ of $B$.

Example: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & a \end{bmatrix}$ and let $B = \begin{bmatrix} -3 & b \\ 1 & -4 \\ 5 & -1 \end{bmatrix}$. What is $AB$?

Example: Let $A = \begin{bmatrix} 7 & 1 & -3 \\ -9 & 8 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} -3 & 2 & 4 \\ 1 & -4 & -6 \end{bmatrix}$. What is $AB$?

You need to know how to multiply matrices by hand, but you can also use the calculator.

Is it true that $AB = BA$? NO, matrix multiplication is NOT commutative.

Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$ and let $B = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$. Find $AB$ and $BA$. 
Matrix multiplication can also be used in word problems. When multiplying matrices where the matrices have meaning, you need to keep two things in mind:

1. Size: Make sure the sizes of the matrices allow you to multiply them.
2. Meaning: Label your matrices, and then make sure the labels on the columns of the first matrix match the labels on the rows of the second matrix. Otherwise, the multiplication doesn’t make sense.

The number of children, students, and adults who attended movie I, II, and III on a given Friday are given in the matrix $A$ below. The admission price is $2 for children, $3 for students, and $4 for adults.

$$A = \begin{bmatrix}
I & II & III \\
225 & 75 & 280 \\
110 & 180 & 85 \\
50 & 225 & 110 \\
\end{bmatrix}$$

Come up with a matrix $B$ to multiply $A$ by where the product matrix will give the amount of money brought in by each movie on that Friday.

There are 4 possibilities:

- $B = \begin{bmatrix}
C & S & A \\
\end{bmatrix}$ and multiply 
- $B = \begin{bmatrix}
C & S \\
\end{bmatrix}$ and multiply 
- $B = \begin{bmatrix}
C \\
\end{bmatrix}$ and multiply 
- $B = \begin{bmatrix}
\end{bmatrix}$ and multiply 

4
Example: The matrix $P$ below gives the number of units of rice, beef, and lamb per serving of three different types of dog food. Matrix $Q$ gives the number of servings of each type of dog food that are fed to three large dogs: Brutus, Max, and Atom.

$P = \begin{pmatrix}
\text{Food I} & \text{Food II} & \text{Food III} \\
\text{rice} & 5 & 9 & 12 \\
\text{beef} & 4 & 8 & 3 \\
\text{lamb} & 7 & 10 & 11
\end{pmatrix}$

$Q = \begin{pmatrix}
\text{Brutus} & \text{Max} & \text{Atom} \\
\text{Food I} & 3 & 4 & 7 \\
\text{Food II} & 5 & 1 & 2 \\
\text{Food III} & 6 & 8 & 3
\end{pmatrix}$

What is the meaning of the product $PQ$?

What is the meaning of the product $QP$?

The Identity Matrix, $I_n$ is the $n \times n$ matrix that has 1’s all along the diagonal and 0’s everywhere else.

For example, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $I_4 =$

The identity matrix for matrices acts like the number 1 for numbers. We know that for any real number $a$, $1 \cdot a = a$ and $a \cdot 1 = a$.

For any matrix $A$, $I_n A = A$ and $A I_n = A$ (assuming the identity matrix is the appropriate size).
5.3 Inverse of a Square Matrix

Let $A$ be a square matrix of size $n$. The inverse of $A$, denoted $A^{-1}$, is the matrix such that $A^{-1}A = AA^{-1} = I_n$.

Verify that the following two matrices are inverses of each other: $A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$.

$$AB = \quad BA =$$

Not all matrices have inverses.

Square matrices that have inverses are said to be **nonsingular**.

Matrices that do NOT have inverses are said to be **singular**.

If a matrix is not square, it CANNOT have an inverse. If a matrix is square, it may or may not have an inverse.

For this class, we will find matrix inverses on the calculator. Just call up the matrix and then press $x^{-1}$.

Find the inverse, if it exists, of each of the following matrices.

$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$; $A^{-1} =$

$B = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 4 & -2 \\ -5 & 0 & -2 \end{bmatrix}$; $B^{-1} =$

We can use inverses to solve matrix equations.

Solve the matrix equation $AX = B$ for $X$.

Solve the matrix equation $XB - E = D$ for $X$. 
Solve the matrix equation $HX + X = F$ for $X$.

If time allows:

There is another way to solve systems of equations using matrices. It involves writing the system as a matrix equation $AX = B$.

Example: Write the following system of linear equations as a matrix equation.

$$
\begin{align*}
2x & - 3y + 4z = 6 \\
2y & - 3z = 7 \\
x & - y + 2z = 4
\end{align*}
$$

We saw before that the solution to the matrix equation $AX = B$ is that $X = A^{-1}B$.

Solve the above system using inverses.

In practice, it is usually much easier to solve a system of equations using rref because a matrix does not always have an inverse.
Additional Honors Topic: Cryptography

Suppose each letter of the alphabet and a space are assigned a number:

<table>
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<tr>
<th>a</th>
<th>b</th>
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</table>

You want to send the message “texas aggies” to your friend.

This message would be encoded as: 20 5 24 1 19 27 1 7 7 9 5 19

However, this is WAY too easy to decode if someone intercepted your message. So, we introduce an encoding matrix $E$. [Any square matrix with positive entries that has an inverse.]

Suppose $E = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 5 \\ 7 & 8 & 10 \end{bmatrix}$.

To code the coded message using this encoder, put the original message in a matrix $M$ and then create the matrix $C = EM$.

We then send the encoded message:

When your friend receives this encoded message, they would have to know the encoding matrix $E$, and then could decode the message by computing $M = E^{-1}C$.

Example: Encode the message “You lose” using the encoding matrix $E = \begin{bmatrix} 1 & 5 \\ 9 & 8 \end{bmatrix}$.

Example: Decode the following message using $E$ above. 52 135 142 279 70 149 32 251 102 215