6.1 Sigma Notation

The Greek letter, \( \sum \), is used to represent a sum of many terms:

If \( a_m, a_{m+1}, \ldots, a_n \) are real numbers and \( m \) and \( n \) are integers with \( m < n \), then

\[
\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n
\]

This form is called sigma notation and the letter \( i \) is called the index of summation.

Example: Calculate \( \sum_{i=2}^{5} (i^2 + i) \).

Example: Calculate \( \sum_{i=1}^{200} 2 \), \( \sum_{i=4}^{200} 2 \), \( \sum_{i=1}^{n} a \)

Example: Write the sum \( \frac{2}{7} + \frac{3}{8} + \frac{4}{9} + \frac{5}{10} + \ldots + \frac{18}{21} \) using sigma notation.

Example: Evaluate \( \sum_{i=5}^{25} \left( \frac{1}{i-1} - \frac{1}{i} \right) \)

Properties of Sums Using Sigma Notation:

(a) \( \sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i \)

(b) \( \sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i \)
Special Sums (Do not memorize):

(a) \( \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \)

(b) \( \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6} \)

(c) \( \sum_{i=1}^{n} i^3 = \left( \frac{n(n + 1)}{2} \right)^2 \)

Calculate \( \sum_{i=1}^{10} i \)

Calculate \( \sum_{i=3}^{10} i^2 \)

Calculate \( \sum_{i=1}^{n} (i - 4)(i + 1) \)

Calculate \( \lim_{n \to \infty} \sum_{i=1}^{n} 2 \left( \frac{2i}{n} + \left( \frac{2i}{n} \right)^3 \right) \)
6.2 Area

Goal: Suppose we have a function $f(x)$ where $f(x) \geq 0$ on the interval $[a, b]$. We want to be able to find the area under the curve between $x = a$ and $x = b$.

We can estimate the area by dividing up the region into intervals and then forming rectangles. The area can then be approximated by the sum of the areas of these rectangles. The more rectangles, the better the approximation.
Method:

1. Determine Width of the Rectangles

Create a **partition** $P$ of the interval $[a, b]$ by dividing the interval into $n$ smaller subintervals.

The $x$-values we choose to divide the interval into subintervals are called the partition numbers and denoted $x_0, x_1, x_2, \ldots, x_n$.  

![Diagram showing a partition with partition numbers $x_0, x_1, \ldots, x_n$]

Notes:

- The first partition number should always be $a$ and the last partition number should always be $b$. However, the subintervals do not have to be equally spaced.
- The lengths of these subintervals will be the widths of our rectangles.
- The length of the $i$th subinterval is denoted $\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$
- The **norm** of the partition, $||P||$, is defined to be the largest $\Delta x_i$. $||P|| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$.

2. Determine Height of the Rectangles

Choose a number within each subinterval $[x_{i-1}, x_i]$. We will call this number $x^*_i$. This number can be the left endpoint, right endpoint, midpoint, or any other point in the subinterval.

We choose the function value at this point, $f(x^*_i)$, to be the height of the rectangle over that interval.

![Graphs showing different heights $f(x^*_i)$ for $i = 1, 2, 3, 4$]

3. Determine Area of the Rectangles

The area of the rectangle corresponding to the subinterval $[x_{i-1}, x_i]$ is now $f(x^*_i)\Delta x_i$.

So, the total area of all the rectangles is $\sum_{i=1}^{n} f(x^*_i)\Delta x_i$. This is called a **Riemann Sum**.
Example: Consider the function $f(x) = 20 - x^2$ on the interval $[0, 4]$. Approximate the area under the curve on this interval by using the partition $P = \{0, 2, 3, 4\}$ and choosing $x_i^*$ to be the left endpoint of each subinterval.

If we want $n$ EQUALLY-SPACED subintervals for an interval $[a, b]$, what is $\Delta x_i$?

Example: Consider the function $f(x) = x^2 + 1$ on the interval $[2, 10]$. Approximate the area under the curve on this interval by using 4 equal-length subintervals and choosing $x_i^*$ to be the midpoint of each subinterval.
Example: Consider the function \( f(x) = x^3 \) on the interval \([0, 3]\). Approximate the area under the curve on this interval by using 6 equal-length subintervals and choosing \( x^*_i \) to be the right endpoint of each subinterval.

**EXACT Area:** These Riemann sums are just an approximation for the area under the curve. As the the widths of these rectangles get smaller and smaller (and we thus have more and more rectangles), we will get closer and closer to the actual area.

\[
A = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x^*_i) \Delta x_i
\]

If the intervals all have the same length, this limit can be expressed as

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i) \Delta x_i
\]

In general then, the exact area under the graph of a curve \( f(x) \) on an interval \([a, b]\) can be found by computing the limit:

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i) \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i \Delta x) \Delta x_i
\]

Note: We could have also used the left endpoint or midpoint instead of the right endpoint to develop the above formula, but it ultimately doesn’t make a difference since we are taking the limit.
Set up the limit to find the exact area under the graph of \( f(x) = \sqrt{x^3 + 1} \) on the interval \([1, 4]\).

Set up the limit to find the exact area under the graph of \( f(x) = \frac{1}{x^2} + \sin(x) \) on the interval \([3, 7]\).
### 6.3 The Definite Integral

We saw in the previous section that if \( f(x) \geq 0 \) on an interval \([a, b]\), then the exact area under the graph between \( x = a \) and \( x = b \) is

\[
A = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

We now define the **definite integral of \( f \)** from \( a \) to \( b \) as the above limit:

\[
\int_{a}^{b} f(x) \, dx = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

If the subintervals are equally spaced, then \( \Delta x = \frac{b - a}{n} \) and this limit can be represented as:

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

If the limit exists, then \( f \) is said to be **integrable** on this interval.

**Notes:** In the notation \( \int_{a}^{b} f(x) \, dx \), \( f(x) \) is called the **integrand** and \( a \) and \( b \) are call the **limits of integration**. (\( a \) is the lower limit and \( b \) is the upper limit).

Once again, if \( f(x) \geq 0 \) on the interval, then the definite integral can be interpreted as the area under the graph of \( f \) between \( x = a \) and \( x = b \)

*If \( f \) is not always positive, the definite integral is still defined, but now represents the **NET** area.*

For the graph of \( f \) below, compute the following definite integrals using the indicated areas.

\[
\int_{0}^{A} f(x) \, dx =
\]

\[
\int_{A}^{B} f(x) \, dx =
\]

\[
\int_{0}^{C} f(x) \, dx =
\]

\[
\int_{A}^{C} f(x) \, dx =
\]

\[
\int_{D}^{C} f(x) \, dx =
\]

How much total area is bounded between the curve and the \( x \)-axis between \( x = 0 \) and \( x = D \)?
Express the following limit as a definite integral on the interval \([0, 2]\).

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \left[ \left( \frac{2i}{n} \right)^6 - 4 \left( \frac{2i}{n} \right) \right]
\]

The **Midpoint Rule** for definite integrals means to approximate the integral by using a midpoint Riemann Sum (just as in 6.2).

Use the Midpoint Rule with \(n = 4\) to approximate \(\int_{-4}^{4} (x^2 - 4) \, dx\).

Evaluate the following definite integrals by interpreting each in terms of area.

(1) \(\int_{-1}^{6} (2x - 4) \, dx\)
(2) \[ \int_0^5 \sqrt{25 - x^2} \, dx \]

(3) \[ \int_{-2}^5 |x - 3| \, dx \]

Properties of the Integral:

(1) \[ \int_a^b c \, dx = c(b - a) \text{ where } c \text{ is any constant} \]

(2) \[ \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \]

(3) \[ \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx \]

(4) \[ \int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx \]

(5) \[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \]

(6) \[ \int_a^a f(x) \, dx = 0 \]

Example: If \[ \int_2^5 f(x) \, dx = 7 \] and \[ \int_2^5 g(x) \, dx = -4 \], calculate \[ \int_2^5 (3f(x) + g(x)) \, dx \].
Example: Write the following as a single integral. \[ \int_{-2}^{3} f(x) \, dx - \int_{-2}^{0} f(x) \, dx + \int_{3}^{5} f(x) \, dx \]

More Properties:

1. If \( f(x) \geq 0 \) for all \( x \) on \([a, b]\), then \( \int_{a}^{b} f(x) \, dx \geq 0 \)

2. If \( f(x) \geq g(x) \) for all \( x \) on \([a, b]\), then \( \int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx \)

3. If \( m \leq f(x) \leq M \) for all \( x \) on \([a, b]\), then
   \[ m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a) \]

Suppose a continuous function \( f(x) \) has an absolute maximum value of 12 and an absolute minimum value of 3 on the interval \([2, 8]\). What can be said about the value of \( \int_{2}^{8} f(x) \, dx \)?

6.4 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 2:

If \( f \) is continuous on \([a, b]\), then
\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

where \( F \) is any antiderivative of \( f \).

Evaluate the following definite integrals.
\[ \int_{1}^{3} \left( \frac{1}{x^3} - \frac{3}{x} \right) \, dx \]
\[
\int_0^2 (3e^x - 2^x) \, dx
\]

\[
\int_1^4 \frac{u^2 - 1}{u \sqrt{u}} \, du
\]

\[
\int_0^{\frac{\pi}{4}} (\sec^2 \theta + 2 \cos \theta) \, d\theta
\]

Find the area under the curve \( f(t) = (t^2 - t)^2 \) between \( t = -2 \) and \( t = 0 \).
Evaluate $\int_{-1/2}^{\pi/2} f(x) \, dx$ where

\[
f(x) = \begin{cases} 
\frac{1}{\sqrt{1-x^2}} & \frac{\pi}{2} \leq x < 0 \\
\sin x & 0 \leq x \leq \frac{\pi}{2}
\end{cases}
\]

Find the area under the curve $f(x) = |x^2 - 4|$ between $x = 0$ and $x = 4$.

If $f(8) = 12$, $f'$ is continuous, and $\int_{-1}^{8} f'(x) \, dx = 9$, what is $f(-1)$?
Application: Suppose an object has velocity function \( v(t) \). Then, on the time interval from \( t = a \) to \( t = b \):

\[
\int_a^b v(t) \, dt = s(b) - s(a)
\]

by the 2nd Part of the Fundamental Theorem since the position function \( s(t) \) is an antiderivative of \( v(t) \). This is the \textit{displacement} of the object on the time interval.

The actual total distance traveled by the object is \( \int_a^b |v(t)| \, dt \). Why?

Example: Suppose the velocity of an object is given by the function \( v(t) = t^2 - 2t - 3 \). Find the displacement and total distance traveled by the object on the time interval \( 0 \leq t \leq 4 \).

The \textbf{indefinite integral} is used to indicate the process of finding the most general antiderivative of \( f(x) \):

\[
\int f(x) \, dx = F(x) + C
\]

where \( F(x) \) is an antiderivative of \( f(x) \) (i.e. \( F'(x) = f(x) \)).

Find the general indefinite integral \( \int \left[ \frac{5}{1 + x^2} + \csc x \cot x - \frac{3}{x} \right] \, dx \)
The Fundamental Theorem of Calculus, Part 1:

If \( f \) is continuous on \( [a, b] \), then the function \( g(x) \) defined by

\[
g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b
\]

is continuous on \( [a, b] \) and differentiable on \( (a, b) \) and \( g'(x) = f(x) \).

In other words,

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

Let \( g(x) = \int_0^x f(t) \, dt \), where \( f \) is graphed below. Calculate \( g(0) \), \( g(2) \), and \( g(6) \). Where is \( g(x) \) increasing? decreasing?

\( g \) represents the “area so far”. The FTC Part 1 says that \( g \) is an antiderivative of \( f \) since \( g'(x) = f(x) \).

Calculate the derivatives of the following functions.

\[
g(x) = \int_1^x t^2 \, dt
\]

\[
g(x) = \int_{\sin x}^{x^3} t^2 \, dt
\]

\[
g(x) = \int_{-7}^x \sqrt{6 - t^2} \, dt
\]

\[
g(x) = \int_x^{12} \sqrt{6 - t^2} \, dt
\]

\[
g(x) = \int_{3}^{x^2} \frac{\ln u}{u^2 + 3} \, du
\]

\[
g(x) = \int_{\sqrt{x}}^{\cos x} \frac{\tan t}{t^2} \, dt
\]