5.5 Applied Maximum/Minimum Problems

Finding extreme values of functions is useful in applications.

When dealing with a max/min problem, recognize what you are trying to maximize/minimize, write this quantity in terms of only ONE variable, and use calculus to find the max/min value of this function. You must show that your answer is a max or min.

Example 1: A rectangular storage container with an open top is to have a volume of \(10 \text{ m}^3\). If the length of the base is three times the width, find the dimensions of the container that will minimize the cost of materials (i.e., minimize the surface area).

\[
V = 10 \\
lwh = 10 \\
(3w)wh = 10 \\
3w^2h = 10 \\
h = \frac{10}{3w^2}
\]

Minimize \(S = lw + 2lh + 2wh\)

\[
S = (3w)w + 2(3w)h + 2wh \\
= 3w^2 + 8wh \\
S = 3w^2 + 8w \left(\frac{10}{3w}\right) \\
S = 3w^2 + \frac{80}{3w} \\
S' = 6w - \frac{80}{3w^2}
\]

\[
S' \text{ DNE when } w = 0 \rightarrow \text{ Not in domain} \\
S' = 0 \text{ when } 18w^2 - 80 = 0 \\
w^2 = \frac{80}{18} = \frac{40}{9} \Rightarrow w = \sqrt[3]{\frac{40}{9}}
\]

**Show** \(w = \sqrt[3]{\frac{40}{9}}\) gives us a min.

<table>
<thead>
<tr>
<th>(w)</th>
<th>(S')</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(\sqrt[3]{\frac{40}{9}})</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>+</td>
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</tbody>
</table>

**Dimensions:** \(w = \sqrt[3]{\frac{40}{9}} \text{ m}, \ l = 3\sqrt[3]{\frac{40}{9}} \text{ m}, \ h = \frac{10}{3\left(\sqrt[3]{\frac{40}{9}}\right)^2} \text{ m}\)
Example 2: (§30, 5.5) A poster is to have a total area of 180 in² consisting of a printed area with a 1-inch border at the bottom and sides and a 2-inch border at the top. What dimensions of the whole poster will give the largest printed area? What is the largest printed area?

Maximize \( A = (x-2)(y-3) \)
\[ A = (x-2)(\frac{180}{x} - 3) \]
\[ A = 180 - 3x - \frac{360}{x} + 6 \]
\[ A = -3x - \frac{360}{x} + 186 \quad , \quad x > 2 \]

\[ A' = -3 + \frac{360}{x^2} = \frac{-3x^2 + 360}{x^2} \]

\( A' \) DNE when \( x = 0 \) \rightarrow Not in domain.
\( A' = 0 \) when \( -3x^2 + 360 = 0 \)
\[ x^2 = 120 \]
\[ x = \sqrt{120} \left( \frac{\sqrt{30}}{2} \right) \]

OR

Show \( x = \sqrt{120} \) gives a max:

2nd Deriv Test.
\[ A'' = -\frac{720}{x^3} \]
\[ A''(\sqrt{120}) = -\frac{720}{(\sqrt{120})^3} < 0 \]

\( A \) is \( CD \) here
\( x = \sqrt{120} \) gives us max.

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\[ x = \frac{180}{\sqrt{120}} \text{ in} \]

Max Area: \(-3\sqrt{120} - \frac{360}{\sqrt{120}} + 186 \text{ in}^2\)
Example 3: Find the point on the line $y = 5x + 2$ that is closest to the point $(4, 1)$.

Minimize $d = \sqrt{(x-4)^2 + (5x+2-1)^2}$

$d = \sqrt{(x-4)^2 + (5x+1)^2}$

$d' = \frac{a(x-4) + a(6x+1)(5)}{\sqrt{(x-4)^2 + (5x+1)^2}}$

$d' = \frac{52x + 2}{2\sqrt{(x-4)^2 + (5x+1)^2}}$

$d' = 0$ when $\text{denom} = 0 \rightarrow \text{never occurs}$

$d' = 0$ when $52x + 2 = 0 \Rightarrow x = -2/52 = -1/26$

Show $x = -1/26$ gives us a min.

$x = -1/26 \Rightarrow y = 5(-1/26) + 2 = \frac{47}{26}$

Min $(-1/26, \frac{47}{26})$
Example 4: Find the dimensions of the largest rectangle that has its base on the $x$-axis and its other two vertices above the $x$-axis lying on the function $f(x) = 11 - x^4$.

Maximize \[ A = 2xy \]

\[ A = 2x(11 - x^4) \]

\[ A = 22x - 2x^5, \quad 0 < x < \sqrt[4]{11} \]

\[ A' = 22 - 10x^4 = 0 \]

\[ 10x^4 = 22 \]

\[ x^4 = \frac{22}{10} = \frac{11}{5} \]

\[ x = \sqrt[4]{\frac{11}{5}} \]

Show $x = \sqrt[4]{\frac{11}{5}}$ gives max:

\[ A'' = -40x^3 \]

\[ A''(\sqrt[4]{\frac{11}{5}}) = -40\left(\sqrt[4]{\frac{11}{5}}\right)^3 < 0 \]

$A$ is CD here $\Rightarrow x = \sqrt[4]{\frac{11}{5}}$ gives a max.

Dimensions: $2x = 2\sqrt[4]{\frac{11}{5}}$ by $y = 11 - \left(\sqrt[4]{\frac{11}{5}}\right)^4 = 11 - \frac{11}{5} = \frac{44}{5}$
Example 5: A 10-m wire is to be cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. What amount of wire (if any) should be used for the circle in order to maximize the total area enclosed? Minimize the total area enclosed.

\[ A = \pi \left( \frac{x}{2\pi} \right)^2 \]

We know \( C = x \)

\[ 2\pi r = x \]

\[ r = \frac{x}{2\pi} \]

So,

\[ A = \pi \left( \frac{x}{2\pi} \right)^2 = \pi \cdot \frac{x^2}{4\pi^2} \]

\[ A = \frac{x^2}{4\pi} \]

Know \( \sin 60^\circ = \frac{\sqrt{3}}{2} \)

So,

\[ h = \frac{\sqrt{3}}{2} \cdot \frac{10-x}{3} \]

\[ A = \frac{1}{2} \left( \frac{10-x}{3} \right) \left( \frac{\sqrt{3}}{2} \right) \left( \frac{10-x}{3} \right) \]

\[ A = \frac{\sqrt{3}}{36} (10-x)^2 \]


Minimize

\[ A = \frac{x^2}{4\pi} + \frac{\sqrt{3}}{36} (10-x)^2, \quad 0 \leq x \leq 10 \]

\[ A' = \frac{x}{2\pi} - \frac{\sqrt{3}}{18} (10-x) \]

\[ A' = \frac{x}{2\pi} - \frac{10\sqrt{3}}{18} + \frac{\sqrt{3} x}{18} = 0 \]

\[ x \left( \frac{1}{2\pi} + \frac{\sqrt{3}}{18} \right) = \frac{10\sqrt{3}}{18} \]

\[ x = \frac{10\sqrt{3}}{18} \approx 3.7679 \]

So, now only CN is \( \approx 3.7679 \):

\[ \frac{3}{7} \quad \text{Min} \]

\[ 0 - 3.7679 + 10 \]

So Abs Min area occurs when \( x = 3.7679 \) m is used for circle. Note: \( A(3.7679) \approx 2.9324 \ m^2 \)

To find when Abs Max area occurs, check endpoints.

\[ A(0) = \frac{\sqrt{3}}{36} \cdot 100 \approx 4.8113 \ m^2 \]

\[ A(10) = \frac{100}{4\pi} \approx 7.9577 \ m^2 \]

Max Area occurs when \( x = 10 \) m used for circle (all the wire) Abs Max area is \( 7.9577 \ m^2 \)
Example 6: A company is designing a closed cylindrical can in which the top and bottom of the can cost $3 per cm² and the side of the can costs $2 per cm². If the company can afford to spend $50 per can, determine the dimensions of the can that would maximize its volume.
5.7 Antiderivatives

Given a function, you are able to take its derivative. But now, we will talk about how to find the original function given its derivative.

Definition: A function \( F \) is called an **antiderivative** of \( f \) on an interval \( I \) if \( F'(x) = f(x) \) for all \( x \) in \( I \).

Find an antiderivative of \( f(x) = 3x^2 \). How many are there?

If \( F \) is an antiderivative of \( f \), then the most general antiderivative of \( f \) is \( F(x) + C \), where \( C \) is an arbitrary constant.

What is the general antiderivative of \( f(x) = x^n \) if \( n \neq -1 \)?

Table of Some Antiderivatives:

<table>
<thead>
<tr>
<th>Function</th>
<th>General Antiderivative</th>
<th>Function</th>
<th>General Antiderivative</th>
</tr>
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<tbody>
<tr>
<td>( a ) (a constant)</td>
<td>( ax + C )</td>
<td>( \frac{1}{x} )</td>
<td>( \log</td>
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<tr>
<td>( x^n ), ( n \neq -1 )</td>
<td>( \frac{x^{n+1}}{n+1} + C )</td>
<td>( \sqrt{1-x^2} )</td>
<td>( \arcsin x + C )</td>
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<tr>
<td>( \frac{1}{x} )</td>
<td>( \ln</td>
<td>x</td>
<td>+ C )</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( e^x + C )</td>
<td>( a^x )</td>
<td>( \frac{a^x}{\ln a} + C )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sin x + C )</td>
<td>( \sec x \tan x )</td>
<td>( \sec x + C )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( -\cos x + C )</td>
<td>( -\csc x \cot x )</td>
<td>( \csc x + C )</td>
</tr>
<tr>
<td>( \sec^2 x )</td>
<td>( \tan x + C )</td>
<td>( -\csc^2 x )</td>
<td>( \cot x + C )</td>
</tr>
</tbody>
</table>

Find the most general antiderivative of \( f(x) = 7x^5 + \sqrt{x} + \frac{3}{\sqrt{x}} + 2 \cos x + \sqrt{x^2} + \frac{9}{1+x^2} \).

Find \( f(x) \) given that \( f'(x) = (1-x^2)^{-1/2} - \sin x + 3 + \sec^2 x \) and \( f(0) = -8 \).
Find $f(x)$ given that $f'(x) = 10 - \frac{2}{x^3} - \frac{6}{x^2} + \frac{3}{x}$ and $f(1) = 7$.

Find $f(x)$ given that $f''(x) = \frac{x + \sqrt{x}}{\sqrt{x}} + e^x$ along with the data $f'(1) = \frac{5}{3}$ and $f(0) = -1$. 
Find \( f(x) \) given that \( f''(x) = \frac{1}{4x\sqrt{x}} \) where \( f(1) = 4 \) and \( f(4) = 12 \).
Application: If we know the velocity function \( v(t) \) of an object, we can now find the position function \( s(t) \), since \( v(t) = s'(t) \). Similarly, given the acceleration function \( a(t) \), we can find both the velocity and position functions.

Example: An object is thrown upward with a speed of 5 m/s from a 400 m observation tower. Find the distance of the stone above the ground at time \( t \). At what time \( t \) does the stone hit the ground, and with what velocity?

Fact: Ignoring air resistance, the acceleration due to gravity is \(-9.8 \text{ m/s}^2\) (or \(-32 \text{ ft/s}^2\)).
A car is traveling at 10 ft/s when the accelerator is pressed, producing a constant acceleration of 30 ft/s². How far does the car travel before reaching a speed of 80 ft/s?
We can also deal with antiderivatives of vector functions in the same way.

Suppose \( \mathbf{a}(t) = (3 \cos t) \mathbf{i} + (5t + 1)^2 \mathbf{j} \). Find the position vector function \( \mathbf{r}(t) \) given that \( \mathbf{v}(0) = -4 \mathbf{j} \) and \( \mathbf{r}(0) = 3 \mathbf{i} + \mathbf{j} \).