5.1 Derivatives and Graphs

What does $f'$ say about $f$?

If $f'(x) > 0$ on an interval, then $f$ is INCREASING on that interval.
If $f'(x) < 0$ on an interval, then $f$ is DECREASING on that interval.

A function has a LOCAL MAXIMUM at $x = a$ if $f(a) \geq f(x)$ for all $x$ "near" $a$. If the derivative changes from positive to negative at $x = a$, then there is a local maximum at $a$ (provided $f$ is continuous at $a$).

A function has a LOCAL MINIMUM at $x = a$ if $f(a) \leq f(x)$ for all $x$ "near" $a$. If the derivative changes from negative to positive at $x = a$, then there is a local minimum at $a$ (provided $f$ is continuous at $a$).

Given the graph of $f'$ below, find the following:

(a) The places where $f$ has a horizontal tangent line.

(b) The intervals on which $f$ is increasing and decreasing.

(c) The values of $x$ at which $f$ has a local maximum or minimum.

\[ f' = 0 \Rightarrow \begin{cases} x = -4, -1, 1, 3 \end{cases} \]

\[ f' + \quad -4 \quad -1 \quad + \quad 1 \quad - \quad 3 \quad + \]

\[ f \text{ increasing on } (-\infty, -4) U (1, 3) \cup (3, \infty) \]
\[ f \text{ decreasing on } (-4, 1) U (1, 3) \]
\[ \text{Local Max at } x = -4, 1 \]
\[ \text{Local Min at } x = -1, 3 \]
What does $f''$ say about $f$?

If $f''(x) > 0$ on an interval, then $f$ is increasing (slopes are becoming bigger), which means $f$ is CONCAVE UP.

If $f''(x) < 0$ on an interval, then $f$ is decreasing (slopes are becoming smaller), which means $f$ is CONCAVE DOWN.

If a function changes concavity at $x = a$, then $f$ has an \text{INFLECTION POINT} at $x = a$ (provided $x = a$ is in the domain of $f$).

Given the graph of $f''$, find the following. Then, sketch a possible graph of $f$.

(a) Intervals where $f$ is increasing or decreasing.
(b) Values of $x$ where $f$ has a local maximum or minimum.
(c) Intervals where $f$ is concave up or concave down.
(d) Values of $x$ where $f$ has an inflection point.

\[\begin{align*}
\text{Min} & \quad \text{Local Max} & \quad \text{Min} \\
-4 & \quad 1 & \quad 3 \\
\text{f'} & \quad \text{f''} & \quad \text{f'''} \\
\text{f inc. on } (-4, 1) \cup (3, \infty) & \quad \text{dec. on } (-\infty, -2) \cup (2, \infty) \\
\text{Local Max at } x = 1 & \quad \text{Local Min at } x = -4, 3 \\
\text{f'''} & \quad \text{f''} & \quad \text{f'} \\
\text{f inc on } (-\infty, -2) \cup (2, \infty) & \quad \text{f dec on } (-2, 2) \\
\text{IPs at } x = -2, 2 & \quad \text{Not IP Min} \\
\end{align*}\]
Sketch a possible graph of a continuous function $f$ that satisfies the following properties:

- $f(-4) = 0$ and $f(3) = -4$
- $f$ has y-intercept $(0, 2)$
- $f'(x) > 0$ on the intervals $(-\infty, -2)$ and $(3, \infty)$
- $f'(x) < 0$ on the interval $(-2, 3)$
- $f''(x) > 0$ on the interval $(-\infty, -4)$
- $f''(x) < 0$ on the interval $(-4, 3) \cup (3, \infty)$
- $\lim_{x \to -\infty} f(x) = -5$
- $\lim_{x \to \infty} f(x) = 4$
Suppose for a graph $f$ that $f'(5) = 0$ and $f''(5) = -3$. What can be said about the graph of $f$ at $x = 5$?

$f$ has a local max at $x = 5$. 

\[ f'' < 0 \]

\[ f \text{ CD} \]
5.2 Maximum and Minimum Values

A function \( f \) has an absolute maximum at \( c \) if \( f(c) \geq f(x) \) for all \( x \) in the domain of \( f \). The function value \( f(c) \) is the maximum value.

A function \( f \) has an absolute minimum at \( c \) if \( f(c) \leq f(x) \) for all \( x \) in the domain of \( f \). The function value \( f(c) \) is the minimum value.

The absolute maximum and minimum values are called the **extreme values** of \( f \).

Example: Find the absolute and local extrema for the following functions.

\[
f(x) = \begin{cases} 
(x+1)^2 & \text{if } -1.5 < x \leq 0 \\
-x^2 - 1 & \text{if } 0 < x \leq 2
\end{cases}
\]

Abs Max: None
Abs Min: -1 (at \( x = 1 \))
Local Max: None
Local Min: -1 (at \( x = 1 \))
If a function has an absolute max or min, they will occur at either a local max/min or at an endpoint of a given interval. So, how do we find the local extrema of a function (without a graph)?

**ALL local extrema of a function \( f \) will occur at places where the derivative is 0 or the derivative is undefined!**

HOWEVER, if \( f'(c) = 0 \) or if \( f'(c) \) does not exist, this does not necessarily mean that there is a local max/min at \( c \). Those places are just the candidates for where there might be a local max/min.
Definition: A critical number of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Every local max/min will occur at a critical number, but again, not every critical number gives rise to local extrema.

Example: Find the critical numbers of the function $f(x) = \sqrt[3]{x^2 - x}$.

\[ f'(x) = \frac{1}{3} (x^2 - x)^{-2/3} \left( 2x - 1 \right) = \frac{2x - 1}{3(x^2 - x)^{2/3}} \]

\[ f'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2} \]

\[ f''(x) \text{ DNE} \Rightarrow \left( x - \frac{1}{2} \right)^{2/3} = 0 \]
\[ x - \frac{1}{2} = 0 \]
\[ x = \frac{1}{2} \]

\[ (N; x = \frac{1}{2}, 0, 1) \]

Example: Find the critical numbers of the function $f(x) = |9 - x^2|$.

\[ f'(x) = 0 \Rightarrow x = 0 \]

\[ f'(x) \text{ DNE} \Rightarrow x = -3, 3 \]
Example: Find the critical numbers of the function $f(x) = x^{3/5}(x+2)^3$. 

$$f'(x) = \frac{2}{5} x^{-3/5} (x+2)^3 + x^{2/5} \cdot 3(x+2)^2$$

$$= \frac{2(x+2)^3}{5x^{3/5}} + \frac{3x^{2/5} (x+2)^2 - 5x^{3/5}}{5x^{3/5}}$$

$$= \frac{2(x+2)^3 + 15x (x+2)^2}{5x^{3/5}} = \frac{(x+2)^2 [2(x+2) + 15x]}{5x^{3/5}} = \frac{(x+2)^2 (17x + 4)}{5x^{3/5}}$$

$f'(x) = 0 \Rightarrow x = -2, -\frac{4}{17}$

$f'(x)$ DNE $\Rightarrow x = 0$
**Extreme Value Theorem:** If a function $f$ is continuous on a closed interval $[a, b]$, then $f$ attains both an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.

The absolute max/min of a continuous function on a closed interval $[a, b]$ will occur at either a local extremum or at an endpoint. To find the absolute max/min, find all critical values, then evaluate the function at the critical values that lie in the interval and at the endpoints of the interval. The max/min of these values IS the absolute max/min.

Example: Find the absolute maximum and minimum values of $f(x) = x^3 - 12x + 1$ on the interval $[-3, 5]$.

1. **Find CN.**
   
   $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2) = 0 \Rightarrow x = 2, -2$

2. **Evaluate original function $f$ at endpoints of interval and at CN that are in interval.**

   Note: Both CN are in interval.
   
   $f(-3) = 10$
   $f(5) = 66 \rightarrow \text{Abs Max}$
   $f(-2) = 17$
   $f(2) = -15 \rightarrow \text{Abs Min}$
Example: Find the absolute maximum and minimum values of \( g(x) = \frac{e^{x^2}}{2x+1} \) on the interval \([0, 3]\). What about on the interval \([-1, 3]\)?

\[
g(x) = \frac{(2x+1)2xe^{x^2} - 2e^{x^2}}{(2x+1)^2} = \frac{2e^{x^2}(2x+1)x - 1}{(2x+1)^2}
\]

1. Find CN:
\[
g'(x) \mid_{x=1/2} = \frac{2e^{1/4}(2x+1)(x-1/2)}{(2x+1)^2} = \frac{2e^{1/4}(2x-1)(x+1)}{(2x+1)^2}
\]

CN: \( x = 1/2 \), not in interval.

2. \( g(0) \) is not continuous on \([0, 3]\).

\[
g(0) = 1, \quad g(3) = e^{1/4} \rightarrow \text{Abs Max}
\]

\[
g(\frac{1}{2}) = e^{1/4} \rightarrow \text{Abs Min}
\]

\* On \([-1, 3]\) \( g \) is not continuous; Method fails.

Example: Find the absolute maximum and minimum values of \( g(x) = x + 2 \sin x \) on the interval \([0, \pi]\).

\[
g(x) = x + 2 \sin x
\]

\[
g'(x) = 1 + 2 \cos x = 0
\]

\[
\cos x = -\frac{1}{2}
\]

\[
x = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots
\]

Only \( \frac{2\pi}{3} \) is in \([0, \pi]\).

\[
g(0) = 0 + 2 \sin 0 = 0 \rightarrow \text{Abs Min}
\]

\[
g\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2(\frac{\sqrt{3}}{2}) = \frac{2\pi}{3} + \sqrt{3} \rightarrow \text{Abs Max}
\]

\[
g(\pi) = \pi + 2(0) = \pi
\]
5.3 Derivatives and the Shapes of Curves

Mean Value Theorem: If $f$ is a differentiable function on the interval $[a, b]$, then there exists a number $c$ between $a$ and $b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Meaning: There is at least one value $c$ where the tangent line at $c$ is parallel to the secant line between $(a, f(a))$ and $(b, f(b))$.

$$m = \frac{f(b) - f(a)}{b-a}$$

$$f'(c_1) = \frac{f(b) - f(a)}{b-a}$$

Given $f(x) = x^3 - 1$ on the interval $[-1, 2]$, show that $f$ satisfies the Mean Value Theorem.

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{7 - (-2)}{3} = 3$$

Where does $f'(x) = 3$?

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

$$\boxed{x = 1}$$
Recall:
If \( f' > 0 \), then \( f \) is increasing. If \( f' < 0 \), then \( f \) is decreasing.

The First Derivative Test: Suppose \( c \) is a critical number of a continuous function \( f \).
(a) If \( f' \) changes from positive to negative at \( c \), then \( f \) has a local maximum at \( c \).
(b) If \( f' \) changes from negative to positive at \( c \), then \( f \) has a local minimum at \( c \).
(c) If \( f' \) does not change sign at \( c \), then \( f \) has no local maximum or minimum at \( c \).

Recall:
If \( f'' > 0 \), then \( f' \) is increasing and \( f \) is concave up.
If \( f'' < 0 \), then \( f' \) is decreasing and \( f \) is concave down.

An inflection point occurs where there is a change in concavity, i.e., \( f'' \) changes sign.

Given the function \( f(x) = x^4 - 12x^3 + 1 \), find any asymptotes, the intervals where \( f \) is increasing and decreasing, identify all local extrema, and identify intervals of concavity and inflection points.

**Domain:** \((-\infty, \infty)\) ; NO Asymptotes

\[
f'(x) = \frac{4x^3 - 36x^2}{x} = 4x^2(x - 9)
\]

Critical Points: \( x = 0, 9 \)

- \( f \) increasing on \((9, \infty)\)
- \( f \) decreasing on \((-\infty, 0) \cup (0, 9)\)
- Local Min at \( x = 9 \)

\[
f''(x) = 12x^2 - 72x = 12x(x - 6)
\]

**Concavity:**
- \( f \) concave up on \((-\infty, 0) \cup (6, \infty)\)
- \( f \) concave down on \((0, 6)\)

**Sign Chart:**
- \( f' \) positive on \((9, \infty)\)
- \( f' \) negative on \((-\infty, 0) \cup (0, 9)\)
- \( f'' \) positive on \((-\infty, 0) \cup (6, \infty)\)
- \( f'' \) negative on \((0, 6)\)
Given the function below and its derivatives, find the domain, any asymptotes, the intervals where \( f \) is increasing and decreasing, identify all local extrema, and identify intervals of concavity and inflection points. Sketch a graph.

\[
f(x) = \frac{2x + 3}{(x-1)^2}, \quad f'(x) = \frac{-2(x + 4)}{(x-1)^3}, \quad f''(x) = \frac{2(2x + 13)}{(x - 1)^4}
\]

[Verify these on your own.]

\[\begin{align*}
D: & \quad x \neq 1 \\
VA: & \quad x = 1 \\
HA: & \quad y = 0 \\
\end{align*}\]

\[
\begin{array}{cccc}
\text{f'}: & (-) & (-) & (-) \\
\text{f''}: & (-) & (+) & (+) & (+) \\
\end{array}
\]

\[
\begin{array}{cccc}
f': & - & 0 & + \\
f: & \bigcup & \bigcap & \bigcup \\
\end{array}
\]

\[
\begin{array}{cccc}
f''': & - & + & + \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{IP:} & & & 1 \\
VA: & 1 & & 1 \\
\end{array}
\]
Given \( f(x) = (2 + 3 \ln x)^2 \), find the intervals where \( f \) is increasing and decreasing and identify any local extrema.

**Domain:** \( (0, \infty) \)

**First Derivative:**

\[
 f'(x) = 2 \left(2 + 3 \ln x\right) \left(\frac{3}{x}\right) = \frac{6 \left(2 + 3 \ln x\right)}{x}
\]

**Critical Points:**

\( f'(x) = 0 \) when \( 2 + 3 \ln x = 0 \)

- \( 3 \ln x = -2 \)
- \( \ln x = -\frac{2}{3} \)
- \( x = e^{-2/3} \)

**Intervals:**

- **Increasing:** \( e^{-2/3}, e^0 \)
- **Decreasing:** \( 0, e^{-2/3} \)

**Local Minimum:**

- \( f \) increasing on \( (e^{-2/3}, \infty) \)
- \( f \) decreasing on \( (0, e^{-2/3}) \)
- Local Minimum at \( x = e^{-2/3} \).
Given \( f(x) = x^2\sqrt{x+3} \), find the intervals where \( f \) is increasing and decreasing and identify any local extrema.

\[ D: \quad x \geq -3 \quad \Rightarrow \quad [-3, \infty) \]

\[
\begin{align*}
   f(x) &= x^2 (x+3)^{1/2} \\
   f'(x) &= 2x (x+3)^{1/2} + x^2 \frac{1}{2} (x+3)^{-1/2} = 2x (x+3)^{1/2} + \frac{x^2}{2(x+3)^{1/2}} \\
   &= \frac{2x(x+3)^{1/2}(x+3)^{1/2} + x^2}{2(x+3)^{1/2}} = \frac{4x(x+3) + x^2}{2(x+3)^{1/2}} \\
   &= \frac{5x^2 + 12x}{2(x+3)^{1/2}}
\end{align*}
\]

Critical Points:

\[ C.N.: \quad x = 0, \quad -\frac{12}{5}, \quad -3 \]

**Sign Chart:**

- \( f' \)
  - \( + \) at \( -\frac{12}{5} \)
  - \( - \) at \( 0 \)
  - \( + \) at \( -3 \)

**Graphical Analysis:**

- \( f \) is increasing on \( (-\infty, -\frac{12}{5}) \) and \( (0, \infty) \)
- \( f \) is decreasing on \( (-\frac{12}{5}, 0) \)
- Local maximum at \( x = 0 \)
- Local minimum at \( x = -\frac{12}{5} \)
Suppose that $f$ is continuous and has a domain all real numbers except $x = -2$. Given $f''(x)$ below, find all inflection points.

$$f''(x) = \frac{(x+2)^3(5-x)}{(x+2)^3}$$

$e^x$ and $3x$ are always positive; $\tan x$ never $= 0$.

$$\begin{array}{c|c|c|c|c|c|c|c}
\text{Interval} & -10 & -6 & -2 & 0 & 5 & 7 & \\
\hline
f'' & (-) & (+) & (-) & (+) & (+) & (+) & \\
\hline
f & + & - & + & + & + & - & \\
\hline
\text{Behavior} & \text{CU} & \text{CD} & \text{CU} & \text{CU} & \text{CU} & \text{CD} & \\
\end{array}$$

Inflection points at $x = -9$ and $x = 5$. 

Nov 12-10:32 PM
The Second Derivative Test: Suppose $f''$ is continuous near $c$.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.

If $f'(c) = 0$ and $f''(c) = 0$, the test is inconclusive, so you would need to use the First Derivative Test.

Given $f(x) = x^3 - x^2$, classify any local extrema using the Second Derivative Test.

$f'(x) = 3x^2 - 2x = x(3x - 2) = 0$

Critical Numbers: $x = 0$, $x = \frac{2}{3}$

$f''(x) = 6x - 2$

$f''(0) = -2 \rightarrow f$ is CD here $\rightarrow x = 0$ gives us local max

$f''(\frac{2}{3}) = 6(\frac{2}{3}) - 2 = 2 \rightarrow f$ is UC here $\rightarrow x = \frac{2}{3}$ gives us a local min.