5.1 Derivatives and Graphs

What does \( f' \) say about \( f \)?

If \( f'(x) > 0 \) on an interval, then \( f \) is INCREASING on that interval.

If \( f'(x) < 0 \) on an interval, then \( f \) is DECREASING on that interval.

A function has a LOCAL MAXIMUM at \( x = a \) if \( f(a) \geq f(x) \) for all \( x \) “near” \( a \). If the derivative changes from positive to negative at \( x = a \), then there is a local maximum at \( a \) (provided \( f \) is continuous at \( a \)).

A function has a LOCAL MINIMUM at \( x = a \) if \( f(a) \leq f(x) \) for all \( x \) “near” \( a \). If the derivative changes from negative to positive at \( x = a \), then there is a local minimum at \( a \) (provided \( f \) is continuous at \( a \)).

Given the graph of \( f' \) below, find the following:

(a) The places where \( f \) has a horizontal tangent line.

(b) The intervals on which \( f \) is increasing and decreasing.

(c) The values of \( x \) at which \( f \) has a local maximum or minimum.
What does $f''$ say about $f$?

If $f''(x) > 0$ on an interval, then $f'$ is increasing (slopes are becoming bigger), which means $f$ is CONCAVE UP.

If $f''(x) < 0$ on an interval, then $f'$ is decreasing (slopes are becoming smaller), which means $f$ is CONCAVE DOWN.

If a function changes concavity at $x = a$, then $f$ has an INFLECTION POINT at $x = a$ (provided $x = a$ is in the domain of $f$.)

Given the graph of $f'$ below, find the following. Then, sketch a possible graph of $f$.

(a) Intervals where $f$ is increasing or decreasing. (b) Values of $x$ where $f$ has a local maximum or minimum.
(c) Intervals where $f$ is concave up or concave down. (d) Values of $x$ where $f$ has an inflection point.
Sketch a possible graph of a continuous function \( f \) that satisfies the following properties:

- \( f(-4) = 0 \) and \( f(3) = -4 \)
- \( f \) has \( y \)-intercept \((0, 2)\)
- \( f'(x) > 0 \) on the intervals \(( -\infty, -2) \) and \((3, \infty)\)
- \( f'(x) < 0 \) on the interval \((-2, 3)\)
- \( f''(x) > 0 \) on the interval \(( -\infty, -4)\)
- \( f''(x) < 0 \) on the interval \((-4, 3) \cup (3, \infty)\)
- \( \lim_{x \to -\infty} f(x) = -5 \)
- \( \lim_{x \to \infty} f(x) = 4 \)

Suppose for a graph \( f \) that \( f'(5) = 0 \) and \( f''(5) = -3 \). What can be said about the graph of \( f \) at \( x = 5 \)?
5.2 Maximum and Minimum Values

A function \( f \) has an absolute maximum at \( c \) if \( f(c) \geq f(x) \) for all \( x \) in the domain of \( f \). The function value \( f(c) \) is the maximum value.

A function \( f \) has an absolute minimum at \( c \) if \( f(c) \leq f(x) \) for all \( x \) in the domain of \( f \). The function value \( f(c) \) is the minimum value.

The absolute maximum and minimum values are called the extreme values of \( f \).

Example: Find the absolute and local extrema for the following functions.

\[
f(x) = \begin{cases} 
(x + 1)^2 & \text{if } -1.5 < x \leq 0 \\
x^2 - 1 & \text{if } 0 < x \leq 2 
\end{cases}
\]

If a function has an absolute max or min, they will occur at either a local max/min or at an endpoint of a given interval. So, how do we find the local extrema of a function (without a graph)?

ALL local extrema of a function \( f \) will occur at places where the derivative is 0 or the derivative is undefined!

However, if \( f'(c) = 0 \) or if \( f'(c) \) does not exist, this does not necessarily mean that there is a local max/min at \( c \). These places are just the candidates for where there might be a local max/min.
Definition: A critical number of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Every local max/min will occur at a critical number, but again, not every critical number gives rise to local extrema.

Example: Find the critical numbers of the function $f(x) = \sqrt[3]{x^2 - x}$.

Example: Find the critical numbers of the function $f(x) = |9 - x^2|$.

Example: Find the critical numbers of the function $f(x) = x^{2/5}(x + 2)^3$. 
Extreme Value Theorem: If a function $f$ is continuous on a closed interval $[a, b]$, then $f$ attains both an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.

The absolute max/min of a continuous function on a closed interval $[a, b]$ will occur at either a local extremum or at an endpoint. To find the absolute max/min, find all critical values, then evaluate the function at the critical values that lie in the interval and at the endpoints of the interval. The max/min of these values IS the absolute max/min.

Example: Find the absolute maximum and minimum values of $f(x) = x^3 - 12x + 1$ on the interval $[-3, 5]$.

Example: Find the absolute maximum and minimum values of $g(x) = \frac{e^{x^2}}{2x + 1}$ on the interval $[0, 3]$? What about on the interval $[-1, 3]$?
Example: Find the absolute maximum and minimum values of $g(x) = x + 2\sin x$ on the interval $[0, \pi]$.

5.3 Derivatives and the Shapes of Curves

**Mean Value Theorem:** If $f$ is a differentiable function on the interval $[a, b]$, then there exists a number $c$ between $a$ and $b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Meaning: There is at least one value $c$ where the tangent line at $c$ is parallel to the secant line between $(a, f(a))$ and $(b, f(b))$.

Given $f(x) = x^3 - 1$ on the interval $[-1, 2]$, show that $f$ satisfies the Mean Value Theorem.
Recall:
If \( f' > 0 \), then \( f \) is \___________. If \( f' < 0 \), then \( f \) is \___________.

The First Derivative Test: Suppose \( c \) is a critical number of a continuous function \( f \).

(a) If \( f' \) changes from positive to negative at \( c \), then \( f \) has a local maximum at \( c \).
(b) If \( f' \) changes from negative to positive at \( c \), then \( f \) has a local minimum at \( c \).
(c) If \( f' \) does not change sign at \( c \), then \( f \) has no local maximum or minimum at \( c \).

Recall:
If \( f'' > 0 \), then \( f' \) is \__________ and \( f \) is \___________.
If \( f'' < 0 \), then \( f' \) is \__________ and \( f \) is \___________.

An inflection point occurs where there is a change in concavity, ie, \( f'' \) changes sign.

Given the function \( f(x) = x^4 - 12x^3 + 1 \), find any asymptotes, the intervals where \( f \) is increasing and decreasing, identify all local extrema, and identify intervals of concavity and inflection points.
Given the function below and its derivatives, find the domain, any asymptotes, the intervals where \( f \) is increasing and decreasing, identify all local extrema, and identify intervals of concavity and inflection points. Sketch a graph.

\[
f(x) = \frac{2x + 3}{(x - 1)^2}, \quad f'(x) = \frac{-2(x + 4)}{(x - 1)^3}, \quad f''(x) = \frac{2(2x + 13)}{(x - 1)^4} \quad \text{[Verify these on your own.]}\]
Given $f(x) = (2 + 3 \ln x)^2$, find the intervals where $f$ is increasing and decreasing and identify any local extrema.

Given $f(x) = x^2 \sqrt{x + 3}$, find the intervals where $f$ is increasing and decreasing and identify any local extrema.
Suppose that \( f \) is continuous and has a domain all real numbers except \( x = -2 \). Given \( f''(x) \) below, find all inflection points.

\[
f''(x) = \frac{e^{-x}(x - 1)^2(x + 9)}{(x + 2)^4(5 - x)e^{4x}}
\]

**The Second Derivative Test:** Suppose \( f'' \) is continuous near \( c \).

(a) If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f \) has a local minimum at \( c \).

(b) If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f \) has a local maximum at \( c \).

If \( f'(c) = 0 \) and \( f''(c) = 0 \), the test is inconclusive, so you would need to use the First Derivative Test.

Given \( f(x) = x^3 - x^2 \), classify any local extrema using the Second Derivative Test.