4.1 Exponential Functions

Exponential functions are those of the form \( f(x) = a^x \) where \( a > 0, a \neq 1 \).

Case 1: \( f(x) = a^x \) where \( a > 1 \)

Increasing Exponential/Exponential Growth

Case 2: \( f(x) = a^x \) where \( 0 < a < 1 \)

Decreasing Exponential/Exponential Decay

Domain: \((-\infty, \infty)\)  
Range: \((0, \infty)\)  
\(y\)-intercept: \((0, 1)\)  
VA: None  
HA: \(y = 0\)  

\[
\lim_{x \to \infty} f(x) = \quad \lim_{x \to -\infty} f(x) = 
\]

Consider exponentials of the form \( f(x) = a^{-x} \).

\[ f(x) = 2^{-x} \]

\[
\lim_{x \to \infty} f(x) = \quad \lim_{x \to -\infty} f(x) = 
\]

\[ f(x) = \left(\frac{1}{3}\right)^{-x} \]

Consider \( f(x) = e^x \) where the base is the natural number \( e \).
Recall the rules of exponents:

\[ (1) \ a^x a^y = a^{x+y} \quad (2) \ \frac{a^x}{a^y} = a^{x-y} \quad (3) \ (a^x)^y = a^{xy} \quad (4) \ (ab)^x = a^x b^x \quad (5) \ \left( \frac{a}{b} \right)^x = \frac{a^x}{b^x} \]

NOTE: \((a + b)^x \neq a^x + b^x\) – DO NOT DO THIS!

Calculate the following limits:

\[ \lim_{x \to -\infty} 0.3^{-x} \quad \lim_{x \to -\infty} 0.3^{-x} \]

\[ \lim_{x \to -3^+} \left(\frac{3}{2}\right)^{\frac{x}{x+3}} = \quad \lim_{x \to -3^-} \left(\frac{3}{2}\right)^{\frac{x}{x+3}} = \]

\[ \lim_{x \to -\infty} \frac{4 + 3e^x}{3 + e^{-x}} \quad \lim_{x \to \infty} \frac{5 + (0.1)^x}{2 + e^{-x}} \]

\[ \lim_{x \to \infty} \frac{3e^{2x} + e^{-3x}}{e^{2x} - 4e^{-3x}} \quad \lim_{x \to -\infty} \frac{3e^{2x} + e^{-3x}}{e^{2x} - 4e^{-3x}} \]
Derivatives of Exponentials:
\[
\frac{d}{dx} e^x = e^x
\]
\[
\frac{d}{dx} e^{g(x)} = g'(x)e^{g(x)}
\]

Calculate the following derivatives:

\[f'(x)\text{ where } f(x) = e^x \sin x + x^2e^{3x} + \frac{1}{e} + \cos(e^x) + x^e + \sqrt{e^{2x} + 1}\]

\[f'(x)\text{ where } f(x) = \frac{e^{-5x}}{1 + e^{-2x}}\]

\[f^{(n)}(x)\text{ where } f(x) = xe^x\]
Find an equation of the tangent line to the curve $y = x^2e^{\sqrt{x}}$ when $x = 4$.

Find the slope of the tangent line to the curve $4e^{xy} - e^x = y$ at the point $(0,3)$.

For what values of $r$ does the function $f(x) = e^{-rx}$ satisfy the differential equation: $2y'' + 3y' - 2y = 0$. 
4.2 Inverse Functions

A function is **one-to-one** if every element in the domain has a UNIQUE value in the range. In other words, if \( f(x_1) = f(x_2) \), then it MUST be true that \( x_1 = x_2 \) if the function is one-to-one.

Show that the function \( f(x) = \frac{x}{x-2} \) is one-to-one.

Horizontal Line Test: Graphically, a function is one-to-one if no horizontal line intersects its graph more than once.

If a function \( f \) is one-to-one, then it has an inverse \( f^{-1} \) defined by:

\[
f^{-1}(y) = x \iff f(x) = y
\]

If \( f \) has domain \( A \) and range \( B \), then \( f^{-1} \) has domain \( B \) and range \( A \).

If \( (a,b) \) is on the graph of \( f \), then \( (b,a) \) is on the graph of \( f^{-1} \).
The graph of $f^{-1}$ is obtained by reflecting the graph of $f$ about the line $y = x$. If $f$ is a continuous function defined on an interval, then $f^{-1}$ is also a continuous function.

Cancellation Equations: If $x$ is in the domain of $f$ and $y$ is in the range of $f$, then

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y$$

Find the inverse of the function $f(x) = \sqrt{2x + 6}$ and state its domain and range.

Find the inverse of the function $f(x) = \frac{4x + 3}{x - 2}$.
If $f$ is a one-to-one differentiable function with inverse $g = f^{-1}$, then $g$ is also differentiable and

$$g'(a) = \frac{1}{f'(g(a))}$$

provided that $f'(g(a)) \neq 0$.

Suppose $f(x) = 3x^5 + 2x^3 - 3$. Find $g'(2)$ where $g$ is the inverse of $f$.

Suppose $f(x) = 4 + 2x + e^x$. Find $g'(5)$ where $g$ is the inverse of $f$. 
4.3 Logarithmic Functions

We’ve dealt with exponential functions and we know that the graph of an exponential function of the form $f(x) = a^x$ is one-to-one, which means it must have an inverse. The inverse of the exponential function $f(x) = a^x$ is the logarithmic function with base $a$.

$$\log_a x = y \iff a^y = x$$

In words, $\log_a x$ is the EXPONENT to which $a$ must be raised to get $x$.

Evaluate the following:

1. $\log_2 32$
2. $\log_3 \frac{1}{81}$
3. $\log_{49} 7$
4. $\log_a 1$
5. Find $x$ such that $\log_4 x = -3$

\[ f(x) = a^x, \ a > 1 \]

$$\begin{align*}
\text{Domain:} & \quad (-\infty, \infty) \\
\text{Range:} & \quad (0, \infty) \\
\text{Asymptotes:} & \quad y = 0 \\
\text{Intercepts:} & \quad (0, 1) \\
\lim_{x \to \infty} a^x & = \infty \\
\lim_{x \to -\infty} a^x & = 0
\end{align*}$$

\[ f(x) = \log_a x, \ a > 1 \]

$$\begin{align*}
\text{Domain:} & \quad \text{(all real numbers)} \\
\text{Range:} & \quad (0, \infty) \\
\text{Asymptotes:} & \quad y = 0 \\
\text{Intercepts:} & \quad (1, 0) \\
\lim_{x \to \infty} \log_a x & = \infty \\
\lim_{x \to 0^+} \log_a x & = -\infty
\end{align*}$$

\[ y \]

\[ x \]

\[ \text{log}_2(x) \quad \text{log}_3(x) \quad \text{log}_4(x) \]
Cancellation equations:
\[ \log_a(a^x) = x \quad \text{for all } x \quad \quad \quad a^{\log_a x} = x \quad \text{for } x > 0 \]

The **common logarithm** is the logarithmic function with base 10. \( \log_{10} x = \log x \)

The **natural logarithm** is the logarithmic function with base \( e \). It has special notation. \( \log_e x = \ln x \)

\[ \ln x = y \iff e^y = x \]

\[ \ln e = 1 \]

\[ \ln(e^x) = x \]

\[ e^{\ln x} = x, \quad x > 0 \]

Calculate the following limits:

1. \( \lim_{x \to 5^+} \log_3(x - 5) \)

2. \( \lim_{x \to \frac{\pi}{2}} \ln(\sin x) \)

3. \( \lim_{x \to 3^-} \log \left( \frac{1}{3 - x} \right) \)

**Properties of Logarithms:**

1. \( \log_a(xy) = \log_a x + \log_a y \)
2. \( \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \)
3. \( \log_a (x^y) = y \log_a x \)

**IMPORTANT:** \( \log_a(x + y) \neq \log_a x + \log_a y \) – **DO NOT** DO THIS

**Examples:**

1. \( \# 12, 4.4 \) Evaluate \( \log_3 10 + \log_3 20 - 3 \log_3 2 \)
(2) Express \( \frac{1}{2} \ln x + b \ln y - c \ln z - \ln(x^2 + 1) \) as a single logarithm.

(3) Calculate \( \lim_{x \to \infty} [\ln(3x^2 + x) - \ln(2x^2 - x)] \)

(4) Solve the equation \( 4^{\log_4 6} + \ln(4x - 2) = 15 \) for \( x \).

(5) Solve the equation \( \log(x - 7) + \log(x + 2) = 1 \) for \( x \).
(6) Solve the equation $3^{2x-9} - 5 = 0$ for $x$.

(7) Find the inverse of the function $f(x) = \frac{2e^{3x-1}}{1 + e^{3x-1}}$.

(8) Find the domain of $y = \log(x^2 - 4)$. 

Change of Base: \( \log_a x = \frac{\ln x}{\ln a} \)

Example: Calculate \( \log_7 12 \) correct to 6 decimal places.

4.4 Derivatives of Logarithmic Functions

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

\[
\frac{d}{dx} \ln |x| = \frac{1}{x}
\]

- **Chain Rule Version:** \( \frac{d}{dx} \ln(g(x)) = \frac{g'(x)}{g(x)} \)

\[
\frac{d}{dx} \log_a x = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}
\]

- **Chain Rule Version:** \( \frac{d}{dx} \log_a(g(x)) = \frac{g'(x)}{g(x) \ln a} \)

Examples: Find the derivatives of the following functions.

(1) \( f(x) = \ln(3x^2 + \sqrt{x}) \)

(2) \( g(x) = \log_2(x^4 - 5x) \)

(3) \( h(x) = x \ln(\cos x) \)
(4) \( f(x) = \sqrt{\ln(x + \ln x)} \)

(5) \( g(t) = \log \left( \frac{t^2 - 4}{t^3 - 7t} \right) \)

Now we have the ability to differentiate exponential functions where the base is not \( e \).

- \( \frac{d}{dx} a^x = a^x \ln a \)

- Chain Rule Version: \( \frac{d}{dx} a^{g(x)} = a^{g(x)}(\ln a)g'(x) \)

Examples: Differentiate the following functions.

(1) \( f(x) = x^3 5^{x^2 - 7x} \)

(2) \( h(\theta) = 4^{\ln(\theta^2 + 1)} \)
Logarithmic Differentiation: Sometimes it is easier to differentiate a function by first taking the logarithm of both sides, differentiating implicitly and then solving for $y'$. Use this method when:
(1) The function is a quotient or product of a lot of terms. – Log. Diff. recommended, but not necessary.
(2) The function is of the form $y = f(x)g(x)$. – Logarithmic Differentiation NECESSARY.

Find $f'(x)$ for $f(x) = \frac{e^{2x} \sqrt{x^5 + 2}}{(x - 1)^4(x^2 + 9)^2}$.

Find $f'(x)$ for $f(x) = x^{\cos x}$.

Find $y'$ for $y = (\sin x)^{e^x}$.