3.8 Higher Derivatives

Higher derivatives involve taking derivatives of derivatives. For example, if \( y = f(x) \), the second derivative is found by taking the derivative of the first derivative.

Notations for the second derivative: \( f''(x), \ y'', \ \frac{d^2y}{dx^2}, \ D^2f(x) \)

Notations for the third derivative: \( f'''(x), \ y''', \ \frac{d^3y}{dx^3}, \ D^3f(x) \)

Notations for the \( n \)th derivative: \( f^{(n)}(x), \ y^{(n)}, \ \frac{d^n y}{dx^n}, \ D^n f(x) \)

Example: Given \( y = x^{10} - 3x^5 + x - 7 \), find \( y''' \) and \( y^{(11)} \)

Example: Find \( f''(x) \) for \( f(x) = (x^2 + 3x)^5 \)

Example: Find \( f^{(43)}(x) \) where \( f(x) = \sin 5x \)
Example: Find a formula for $f^{(n)}(x)$ where $f(x) = \frac{1}{x - 1}$.

We have seen that the derivative of position is velocity: If $s(t)$ is the position function, then $v(t) = s'(t)$.

The instantaneous rate of change (derivative) of the velocity is called acceleration.

$$a(t) = v'(t) = s''(t) \quad \text{OR} \quad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Example: The position of a particle moving in a straight line is $s(t) = 2t^3 - 7t^2 + 4t + 1$ where position is measured in ft and time is measured in seconds.

(a) Find the velocity and acceleration functions.

(b) How fast is the velocity changing at $t = 1$ seconds?

(c) What is the acceleration at the points where the object is at rest?
If motion is 2-dimensional with position vector function \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \), then \( \mathbf{a}(t) = \langle x''(t), y''(t) \rangle \).

Example: Given the position function \( \mathbf{r}(t) = \langle \sqrt{t^2 + 5}, t^2 + 2t \rangle \), find the acceleration at \( t = 2 \).

Find \( y'' \) for the equation \( x^3 + 3y^2 = 1 \).
3.9 Slopes and Tangents to Parametric Curves

Consider \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) and its derivative at \( t = a \), \( \mathbf{r}'(a) = \langle x'(a), y'(a) \rangle \). This vector is a tangent vector to the graph at the point where \( t = a \). What would be the slope of the tangent line?

So, without initially eliminating the parameter, we can find the slope of the tangent line, \( \frac{dy}{dx} \), by using

\[
\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{dy}{dx} \\
\]

Example: Find an equation of the tangent line to the curve \( \mathbf{r}(t) = \langle t^2 + t, \sqrt{t} \rangle \) when \( t = 4 \).

Example: What is the slope of the tangent line to the curve \( x = 3t + 8, y = 5t^2 + t \) at the point \( (2, 18) \).

Example: Find an equation of the tangent line to the curve \( \mathbf{r}(t) = \langle 5 \cos t, -10 \sin t \rangle \) at the point \( (3, 8) \).
A curve has a horizontal tangent where:

A curve has a vertical tangent where:

Example: Find all points on the curve defined by the parametric equations \(x = t^3 - 12t\) and \(y = 2t^3 - 9t^2\) where the tangent line is horizontal or vertical.

Example: Find both equations of the tangent lines to the curve \(x = \sin 2t, y = 4 \sin t\) at \((0, 0)\). Then find the values of \(t\) for which the tangent line is horizontal or vertical.
3.10 Related Rates

Related rate word problems deal with finding the rate of change (derivative) of one quantity in terms of the rate of change of some other related quantity.

Strategy for Related Rate Word Problems:

(1) Read the problem and write down what you know and what you are trying to find. Draw a picture if needed.

(2) Find an equation that relates the known quantities to the unknown. This may require rewriting one variable in terms of another.

(3) Take the derivative of both sides with respect to \( t \). Plug in the known values and solve for the unknown.

Examples: (I would also suggest looking at the worked-out examples in the textbook for more.)

1. A circular area of snow is melting so that the radius is decreasing at a rate of 3 inches/day. At what rate is the area of the circle changing when the radius of the circle is 12 inches?

2. (#29, Section 3.10) A 10-ft ladder rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is \( \frac{\pi}{4} \) radians?

At what rate is the angle changing when the base of the ladder is 7 ft from the wall?
3. A baseball diamond is a square with 90 ft sides. A batter hits the ball and runs toward first base with a speed of 8 ft/s.

(a) At what rate is his distance from third base increasing when he is halfway to first base?

(b) Suppose at the same time the batter begins to run for first base, a man on third base begins to run for home plate at a speed of 7 ft/s. At what rate is the distance between the two men changing 4 seconds later?
4. (# 8, Section 3.10) A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, at what rate is his shadow on the building changing when he is 4 m from the building?

5. Two sides of a triangle have lengths 8 ft and 5 ft. Suppose the angle between these two sides is increasing at a rate of \( \frac{\pi}{12} \) radians/min. How fast is the area of the triangle increasing at the moment when the angle between the two sides is \( \frac{\pi}{6} \).

6. (#13, Section 3.10) At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00pm?
7. Water is being pumped into an inverted circular cone with radius 4 m and height 10 m at a rate of 5 m$^3$/min. At what rate is the water level rising when the water is 3 m deep?
8. A trough is 20 ft long and has ends which are isosceles triangles that are 4 ft across the top with a height of 5 ft. Suppose water is leaking out of the trough at a rate of \(4 \text{ ft}^3/\text{hr}\) at the same time that water is being pumped in at some constant rate. If the water level is rising at a rate of 1 ft/hr when the height of the water is 2 ft, find the rate at which water is being pumped in.
3.11 Linear and Quadratic Approximations

The equation of the tangent line to the graph of \( f(x) \) at the point \((a, f(a))\) can be used to approximate values of \( f \) that are near \( a \).

This is called the **linear approximation** or **tangent line approximation** of \( f \) at \( a \) and the function

\[
L(x) = f(a) + f'(a)(x - a)
\]

is called the **linearization** of \( f \) at \( a \).

Example: Find the linearization \( L(x) \) of the function \( f(x) = \sqrt{x} \) near \( a = 25 \) and use it to approximate \( \sqrt{24.99} \).

Example: Use a linear approximation to find an approximate value for \( 16.2^{3/4} \).

Example: Use a linear approximation to find an approximate value for \( \cos(29^\circ) \).
Example: For a given function, we know that \( f(7) = 4 \) and that \( f'(x) = \frac{1}{x^2 - 40} \) for all \( x \). Use a linear approximation to estimate \( f(7.05) \).

Example: Suppose for a function \( f(x) \) that the linear approximation at \( a = 5 \) is given by the tangent line \( y = 2x - 1 \).

(a) What are \( f(5) \) and \( f'(5) \)?

(b) If \( g(x) = \frac{1}{\sqrt{f(x)}} \), find the linear approximation for \( g(x) \) at \( a = 5 \).

Example: Find the linearization \( L(x) \) of the function \( f(x) = (x + 3)^2 \) at \( a = 2 \) and use it to approximate the value of \( 4.9^2 \).
Example: Find the linearization $L(x)$ of the function $f(x) = \sqrt{11 - x}$ at $a = 3$ and use it to approximate $\sqrt{7.98}$.

**Quadratic Approximations:** Sometimes linear approximations are not the best and a second-degree (quadratic) approximation is better suited. In this case, we are approximating using a parabola instead of a line.

The quadratic approximation to $f(x)$ near $a$ is

$$P(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Example: Find the quadratic approximation to $f(x) = \sin x$ near $a = \frac{\pi}{2}$.

Example: Find the quadratic approximation to $f(x) = (2x + 5)^5$ near $a = -2$. 