3.5 The Chain Rule

To find the derivative of a function that is the composition of two functions for which we already know the derivatives, we can use the Chain Rule.

The Chain Rule: Suppose $F(x) = f(g(x))$. Then, provided that $g'(x)$ and $f'(g(x))$ both exist,

$$F'(x) = f'(g(x))g'(x)$$

Alternate way of thinking about it: If $y = f(u)$ and $u = g(x)$ where both are differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Find the derivatives of the following functions.

- $f(x) = \sqrt{x^2 + 9} = (x^2 + 9)^{1/2}$

$$f'(x) = \frac{1}{2} (x^2 + 9)^{-1/2} \cdot 2x = \frac{2x}{2\sqrt{x^2 + 9}} = \frac{x}{\sqrt{x^2 + 9}}$$
\[
\frac{(3t^2 + 2t^{-4})^5}{g(t) = \left(3t^2 + \frac{2}{t^4}\right)^5 (2t - \cos t)^7}
\]
\[
g'(t) = 5 \left(3t^2 + 2t^{-4}\right)^4 \left(6t - 8t^{-5}\right) (2t - \cos t)^7 + \left(3t^2 + 2t^{-4}\right)^5 \cdot 7 (2t - \cos t)^6 \cdot (2 + \sin t)
\]
\[
f(x) = \left(\frac{5x - 7}{4x^3 - x}\right)^{10}
\]
\[
f'(x) = 10 \left(\frac{5x - 7}{4x^3 - x}\right)^9 \cdot \left[\frac{(4x^2 - x)(5) - (5x - 7)(12x^2 - 1)}{(4x^3 - x)^2}\right]
\]
\[
g(x) = \sqrt[4]{5x^4 + \sqrt{6x + 1}} = \left(5x^4 + \sqrt[4]{6x + 1}\right)^{1/4}
\]
\[
g'(x) = \frac{1}{4} \left[5x^4 + (6x + 1)^{1/2}\right]^{-3/4} \cdot (20x^3 + \frac{1}{2}(6x + 1)^{-1/2} \cdot 6)
\]
\[ f(x) = \tan(5x) \sec 3x + \cos(x^3) + \sin^2 x + \cos^4 2x \]

\[ \sin^2 x = (\sin x)^2 \]

\[ \cos^4 2x = (\cos 2x)^4 \]

\[ 5 \sec^2 (5x) \sec 3x + \tan(5x) \cdot 3 \sec (3x) \tan (3x) \]

\[ -\sin (x^3) \cdot 3x^2 \]

\[ 2 \sin x \cdot \cos x \]

\[ 4 \left( \cos 2x \right)^3 \cdot (-2 \sin 2x) \]

\[ f(x) = \tan^3 (\csc 4x) = \left[ \tan (\csc 4x) \right]^3 \]

\[ f'(x) = 3 \left[ \tan (\csc 4x) \right]^2 \cdot \sec^2 (\csc 4x) \cdot (-4 \csc 4x \cot 4x) \]

\[ h(x) = \sec (\sin x) \]

\[ h'(x) = \sec (\sin x) \tan (\sin x) \cdot \cos x \]
• Suppose that \( F(x) = f(g(x)) \) where \( f'(4) = 3 \), \( f'(5) = 2 \), \( g'(4) = 6 \) and \( g(4) = 5 \).

(a) Calculate \( F'(4) \).

\[
F'(x) = f'(g(x)) \cdot g'(x)
\]
\[
F'(4) = f'(g(4)) \cdot g'(4)
= f'(5) \cdot g'(4)
= (2) \cdot (6) = 12
\]

(b) Calculate \( G'(0) \) where \( G(x) = f(\sin x + 5) + (g(5x + 4))^2 \)

\[
G'(x) = f'(\sin x + 5)(\cos x) + 2g(5x+4)g'(5x+4) \cdot 5
\]
\[
G'(0) = f'(5) \cdot 1 + 2g(4)g'(4) \cdot 5
= 2 + 2(5)(6)(5)
= 150
\]
3.6 Implicit Differentiation

Consider the equation $y^3 + 2xy = 9$. How would you calculate $\frac{dy}{dx}$?

When $y$ cannot be written explicitly as a function of $x$ (or not easily), we can use the method of implicit differentiation.

To find $\frac{dy}{dx}$, differentiate both sides with respect to $x$, remembering that the Chain Rule is necessary since $y$ is dependent on $x$.

Find $\frac{dy}{dx}$ for the equation $x^2 + y + 3y^2 = 16$

\[
2x + \frac{dy}{dx} + 6y \cdot \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} + 6y \cdot \frac{dy}{dx} = -2x
\]

\[
\frac{dy}{dx} \left[ 1 + 6y \right] = -2x
\]

\[
\frac{dy}{dx} = \frac{-2x}{1 + 6y}
\]
Find \( y' \) if \((y^2 + 1)^3 \). What is the slope of the tangent line at the point \((2, 1)\)?

\[
\frac{3(y^2 + 1)^2 \frac{dy}{dx} (2y + \frac{dy}{dx})}{\frac{dy}{dx} [6y(y^2 + 1)^2 + x + 2]} + x \cdot \frac{dy}{dx} = 6x - 2 \frac{dy}{dx}
\]

\[
\frac{dy}{dx} = \frac{6x - y}{6y(y^2 + 1)^2 + x + 2}
\]

\[
\frac{dy}{dx} \bigg| _{(x,y) = (2,1)} = \frac{11}{28}
\]

Find \( \frac{dy}{dx} \) if \( x^2 - 5x^4y^2 = 4y^2 \).

\[
2x - [20x^3y^2 + 5x^4 \cdot 2y \cdot \frac{dy}{dx}] = 8y \cdot \frac{dy}{dx}
\]

\[
2x - 20x^3y^2 = \left[ 8y + 10x^4y \right] \frac{dy}{dx}
\]

\[
\frac{2x - 20x^3y^2}{8y + 10x^4y} = \frac{dy}{dx}
\]

Find \( \frac{dx}{dy} \) if \( x^2 - 5x^4y^2 = 4y^2 \).

\[
2x \cdot \frac{dx}{dy} - [20x^3 \frac{dx}{dy} y^2 + 5x^4 \cdot 2y] = 8y
\]

\[
\frac{dx}{dy} \cdot [2x - 20x^3y^2] = 8y + 10x^4y
\]

\[
\frac{dx}{dy} = \frac{8y + 10x^4y}{2x - 20x^3y^2}
\]
Find \( \frac{dy}{dx} \) where \( \sin(x-y) + \cos 2y = y \cos x \).

\[
\cos(x-y) \left( 1 - \frac{dy}{dx} \right) - \sin 2y \cdot 2 \frac{dy}{dx} = \frac{dy}{dx} \cos x + y (-\sin x)
\]

\[
\cos(x-y) - \cos(x-y) \frac{dy}{dx}
\]

\[
\frac{dy}{dx} \left[ -\cos(x-y) - 2\sin 2y - \cos x \right] = -y \sin x - \cos (x-y)
\]

\[
\frac{dy}{dx} = \frac{-y \sin x - \cos (x-y)}{-\cos(x-y) - 2\sin 2y - \cos x}
\]

Example: Find an equation of the tangent line to the hyperbola \( \frac{y^2}{36} - \frac{x^2}{4} = 1 \) at the point \((1, -3\sqrt{5})\).

\[
\frac{1}{36} y^2 - \frac{1}{4} x = 1
\]

\[
\frac{1}{18} y \cdot \frac{dy}{dx} - \frac{1}{2} x = 0
\]

\[
\frac{18}{y} \cdot \frac{dy}{dx} = \frac{x}{2} \cdot \frac{18}{y}
\]

\[
\frac{dy}{dx} = \frac{18x}{2y} = \frac{9x}{y}
\]

\[
m \at (1, -3\sqrt{5}) = \frac{9(1)}{-3\sqrt{5}} = -\frac{3}{\sqrt{5}}
\]

\[
y + 3\sqrt{5} = -\frac{3}{\sqrt{5}} (x - 1)
\]
The curves \( x^2 - y^2 = 5 \) and \( 4x^2 + 9y^2 = 72 \) intersect at the point \((3, -2)\). Show that the tangent lines to the two curves at this point are orthogonal.

\[
\begin{align*}
x^2 - y^2 &= 5 \\
2x - 2y \frac{dy}{dx} &= 0 \\
-2y \frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= \frac{2x}{-2y} = \frac{x}{-y}
\end{align*}
\]

\[
\begin{align*}
4x^2 + 9y^2 &= 72 \\
8x + 18y \frac{dy}{dx} &= 0 \\
18y \frac{dy}{dx} &= -8x \\
\frac{dy}{dx} &= \frac{-8x}{18y} = \frac{-4x}{9y}
\end{align*}
\]

At \((3, -2)\): \( m = \frac{3}{-2} \)

At \((3, -2)\): \( m = \frac{-4(3)}{9(-2)} = \frac{-12}{18} = -\frac{2}{3} \)

Since slopes are negative reciprocals, tangent lines are orthogonal.

Definition: Two curves are called orthogonal if at every point of intersection, the tangent lines at those points are orthogonal.

The above curves also intersect at \((3, 2)\), \((-3, 2)\), and \((-3, -2)\). At each of these points, the tangent lines are orthogonal, so these curves are orthogonal.

Two families of curves are orthogonal trajectories of each other if every curve of one family is orthogonal to every curve in the other family.

Show that the families of curves \( y = ax^3 \) and \( x^2 + 3y^2 = b \) are orthogonal trajectories of each other.
3.7 Derivatives of Vector Functions

If \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) is a vector function, then \( \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle \) if both of these derivatives exist.

Recall that the derivative at \( a \), \( \mathbf{r}'(a) \), is the tangent vector to the curve when \( t = a \). However, it also represents the instantaneous velocity of an object with position function \( \mathbf{r}(t) \). So, the tangent vector (or velocity vector) points in the direction of motion as \( t \) increases.

Example: Find the domain of the vector function \( \mathbf{r}(t) = \left( \frac{t}{t+4}, \sqrt{36-t^2} \right) \).

\[ t \neq 0, \quad 36 - t^2 \geq 0, \quad t \leq 6 \]

\[ \left[ \frac{3}{2}, 6 \right) \cup (6, \infty) \]

Find the tangent vector to the curve at the point where \( t = 3 \).

\[ \mathbf{r}'(t) = \left( \frac{t-6}{(t+6)^2}, \frac{1}{2} \right), \quad \mathbf{r}'(3) = \left( \frac{-6}{(9)^2}, \frac{3}{2} \right) = \left( -\frac{6}{81}, \frac{3}{2} \right) \]

Find parametric equations for the tangent line at this point.

\[ \mathbf{T}(s) = \mathbf{r}_0 + s \mathbf{v} \]

\[ \mathbf{v} = \langle -\frac{2}{3}, \frac{3}{4} \rangle \]

\[ \mathbf{r}_0 = \mathbf{r}(3) = \langle \frac{2}{3}, \sqrt{4} \rangle = \langle -1, 2 \rangle \]

\[ \mathbf{T}(s) = \langle -1, 2 \rangle + s \langle -\frac{2}{3}, \frac{3}{4} \rangle \]

\[ x = -1 + \frac{2}{3}s \]

\[ y = 2 + \frac{3}{4}s \]
Example: Find a unit tangent vector to the curve \( \mathbf{r}(t) = \langle t \sin t, 4 - 2 \cos 3t \rangle \) at the point where \( t = \frac{\pi}{3} \).

\[
\mathbf{r}'(t) = \langle \sin t + t \cos t, -2(3 \sin 3t) \rangle \\
= \langle \sin t + t \cos t, 6 \sin 3t \rangle \\
\mathbf{r}'\left(\frac{\pi}{3}\right) = \langle 1 + \frac{\pi}{3} \cdot 0, 6 \sin \frac{3\pi}{2}\rangle = \langle 1, -6 \rangle
\]

\[
\frac{\langle 1, -6 \rangle}{|\langle 1, -6 \rangle|} = \frac{\langle 1, -6 \rangle}{\sqrt{37}} = \langle \frac{1}{\sqrt{37}}, \frac{-6}{\sqrt{37}} \rangle
\]

Find a unit tangent vector to the curve \( \mathbf{r}(t) = \langle 5t^2 + 1, 8t^2 - t \rangle \) at the point \((6, 9)\).

\[
\mathbf{r}'(t) = \langle 10t, 16t - 1 \rangle \\
\mathbf{r}'(-1) = \langle -10, -17 \rangle
\]

\[
\frac{\langle -10, -17 \rangle}{\sqrt{10^2 + 17^2}} = \langle \frac{-10}{\sqrt{369}}, \frac{-17}{\sqrt{369}} \rangle
\]

What is \( t \) at the point \((6, 9)\)?

\[
5t^2 + 1 = 6 \quad \text{AND} \quad 8t^2 - t = 9
\]

\[
5t^2 = 5 \quad \text{AND} \quad t = 1; \quad 8 - 1 \neq 9
\]

\[
t = -1; \quad 8 + 1 = 9 \quad \checkmark
\]

\[
t = -1
\]

\[
8t^2 - t - 9 = 0 \\
(8t - 9)(t + 1) = 0 \\
t = \frac{9}{8}, -1
\]
Again, if \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) is vector function representing the position of a particle at time \( t \), then the tangent vector \( \mathbf{r}'(t) \) is the instantaneous velocity at time \( t \) and the instantaneous speed at time \( t \) is \( |\mathbf{r}'(t)| \).
(Velocity = Vector; Speed = Scalar)

Example: A projectile is fired so that its position is given by the function \( \mathbf{r}(t) = \langle 48\sqrt{3}t, 48t - 16t^2 \rangle \).
Find the velocity and speed at time \( t = 2 \).

\[
\mathbf{v}(t) = \mathbf{r}'(t) = \langle 48\sqrt{3}, 48 - 32t \rangle
\]

\[
\mathbf{v}(2) = \langle 48\sqrt{3}, 48 - 32(2) \rangle = \langle 48\sqrt{3}, -16 \rangle
\to \text{velocity}
\]

\[
\text{speed} = |\mathbf{v}(2)| = \sqrt{(48\sqrt{3})^2 + 16^2}
\]

What is the velocity when the projectile hits the ground?

Occurs when \( y \)-comp of position = 0.

\[
48t - 16t^2 = 0
\]

\[
16t(3 - t) = 0
\]

\[
t = 0, \ t = 3
\]
To find the angle of intersection between two curves, find the angle between the tangent vectors at the point of intersection.

Example: Find the angle of intersection of the curves \( \mathbf{r}_1(t) = <4-t, t^2-5> \) and \( \mathbf{r}_2(s) = <\sqrt{s-1}, s+2> \) if it is known the curves intersect at the point \((1,4)\).

\[ \mathbf{r}_1'(t) = <-1, 2t> \quad \mathbf{r}_2'(s) = <\frac{1}{2} (s-1)^{-\frac{1}{2}}, 1> \]

What are \( t \) and \( s \) at \((1,4)\)?

\[
\begin{align*}
4-t &= 1 \quad \text{AND} \quad t^2-5 &= 4 \\
t &= 3
\end{align*}
\]

\[
\begin{align*}
\sqrt{s-1} &= 1 \quad \text{AND} \quad s+2 &= 4 \\
s &= 2
\end{align*}
\]

Tangent Vectors:
\[ \mathbf{r}_1'(3) = <-1, 6> \quad \mathbf{r}_2'(2) = <\frac{1}{2}, 1> \]

\[
\cos \theta = \frac{<-1, 6> \cdot <\frac{1}{2}, 1>}{|<-1, 6>| \cdot |<\frac{1}{2}, 1>|} = \frac{-\frac{1}{2} + 6}{\sqrt{37} \sqrt{\frac{1}{4} + 1}}
\]

\[
\theta = \cos^{-1}\left(\frac{-\frac{1}{2} + 6}{\sqrt{37} \cdot \sqrt{\frac{5}{4}}} \right) \approx 36.0274^\circ
\]