Chapter 4: Exponential and Logarithmic Functions

4.1 Exponential Functions

The exponential function with base \( a \) is defined for all real numbers \( x \) by

\[
f(x) = a^x
\]

where \( a > 0 \) and \( a \neq 1 \).

We say \( a \neq 1 \), because if \( a = 1 \), then \( a^x = 1^x = 1 \), which is just a constant function (a horizontal line).

Sketch the graph of \( f(x) = 3^x \).

Sketch the graph of \( f(x) = \left(\frac{1}{3}\right)^x \).

Properties of exponential functions of the form \( f(x) = a^x \):

- Domain:
- Range:
- \( y \)-intercept:
- Horizontal asymptote:
- If \( a > 1 \), the graph is ___________. If \( 0 < a < 1 \), the graph is ___________.

Find the exponential function \( f(x) = a^x \) whose graph passes through the point \((3, 8)\).
We can find graphs of other exponential functions by transforming the graphs of these basic exponential functions.

Sketch the graph of \( f(x) = -3^{x-1} \). What are the domain and range of \( f \)? What is the horizontal asymptote?

Sketch the graph of \( f(x) = 2^{-x} + 1 \). What are the domain and range of \( f \)? What is the horizontal asymptote?

Find the function of the form \( f(x) = Ca^x \) whose graph passes through the point \((2, 36)\) and has a \( y \)-intercept of \((0, 4)\).

The **natural exponential function** is the exponential function

\[
    f(x) = e^x
\]

with base \( e \). This is *the* exponential function.
Application: Compound Interest is calculated by the formula

\[ A(t) = P \left(1 + \frac{r}{n}\right)^{nt} \]

where \( A(t) \) = amount after \( t \) years, \( P \) = Principal (original amount), \( r \) = interest rate per year as a decimal, \( n \) = number of times interest is compounded per year, \( t \) = number of years

Example: An investment of $1000 is put into an account with an interest rate of 3%/yr. How much money is in the account after 5 years if interest is compounded monthly? weekly?

Continuously compounded interest is calculated by the formula

\[ A(t) = Pe^{rt} \]

How much money is in the same account above if interest is compounded continuously?

4.2 Logarithmic Functions

Every exponential function is 1-1 by the horizontal line test. Thus, every exponential function has an inverse.

The inverse function of \( f(x) = a^x \) is the logarithmic function \( f^{-1}(x) = \log_a x \).

Let \( a \neq 1 \). The logarithmic function with base \( a \), denoted \( \log_a \), is defined by

So, \( \log_a x \) is the exponent to which \( a \) must be raised to get \( x \).

\[
\begin{align*}
\log_2 8 &= 3 \text{ since } 2^3 = 8 \\
\log_3 \frac{1}{9} &= -2 \text{ since } 3^{-2} = \frac{1}{9} \\
\log_9 3 &= \frac{1}{2} \text{ since } 9^{1/2} = 3 \\
\log_a x &= y \text{ is logarithmic form and } a^y = x \text{ is exponential form. They are equivalent statements.}
\end{align*}
\]

Evaluate the following expressions.

- \( \log_2 32 \)
- \( \log_{64} 4 \)
- \( \log_9 \sqrt{3} \)
Properties of Logarithms

1. \( \log_a 1 = 0 \)
2. \( \log_a a = 1 \)
3. \( \log_a a^x = x \)
4. \( a^{\log_a x} = x \)

What is \( \log_3 3^2 \)?

What is \( 4^{\log_4 17} \)?

Examples. Solve for \( x \) in the following logarithmic equations.

- \( \log_5 x = -4 \)

- \( \log_x 1000 = 3 \)

- \( \log_x 6 = \frac{1}{2} \)

Sketch the graph of \( f(x) = \log_2 x \).

Note that since the inverse of an exponential function is a logarithmic function, then their graphs are reflections of each other across the line \( y = x \).

Properties of logarithmic functions of the form \( f(x) = \log_a x \):

- Domain:
- Range:
- \( x \)-intercept:
- Vertical asymptote:
Find the logarithmic function \( f(x) = \log_a x \) that passes through the point \((9, 2)\).

As with exponential functions, we can find the graphs of other logarithmic functions by transforming the graphs of these basic logarithmic functions.

Sketch the graph of \( f(x) = -\log_2(x - 4) \). What are the domain and range of \( f \)? What is the vertical asymptote?

Sketch the graph of \( f(x) = \log_3(-x) + 1 \). What are the domain and range of \( f \)? What is the vertical asymptote?

The logarithm with base 10 is called the **common logarithm** and is denoted by just writing \( \log x \) instead of \( \log_{10} x \).

Evaluate \( \log 1000 \).

Solve \( \log x = -2 \) for \( x \).

The logarithm with base \( e \) is called the **natural logarithm** and is denoted by \( \ln x \). (So \( \ln x = \log_e x \).) The natural logarithm \( \ln x \) is the inverse function of the exponential function \( e^x \).

Properties of Natural Logarithms

1. \( \ln 1 = 0 \)
2. \( \ln e = 1 \)
3. \( \ln e^x = x \)
4. \( e^{\ln x} = x \)

Evaluate: \( \frac{1}{e^4} \) \( e^{\ln 12} \)
We already saw that the domain of a logarithmic function $\log_a x$ is $(0, \infty)$. In general, this means that we can only take the log of a positive number.

Find the domain of the function $f(x) = \log_5 (8 - 2x)$.

Find the domain of the function $f(x) = \ln(x - x^2)$.