Chapter 2, Continued

2.6 Modeling with Functions

This section is very similar to 1.6, except now we are using functions. When modeling functions,

1. Choose the variable. Sometimes the problem will tell you what to let your variable be.
2. Write all other unknowns in terms of your chosen variable.
3. Set up the model by writing the function in terms of the variable.
4. Interpret and use the model to answer questions.

Examples:

1. (2.6 #4) The height of a cylinder is 3 times its radius. Find a function that models the volume $V$ of the cylinder in terms of its radius, $r$.

2. (2.6 # 20) Find two positive numbers whose sum is 100 and the sum of whose squares is a minimum.
3. (2.6 # 23) A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not plan on building a fence along the river.

(a) Find a function that models the area of the field in terms of one of the sides perpendicular to the river.

(b) Find the dimensions that of the largest area that he can fence.

4. (modified from 2.6 # 31) A box with a square base is to have a volume of 12 ft$^3$. Find a function that models the surface area $S$ of the box in terms of the length $x$ of one side of its base. What are the dimensions that minimize the surface area?
2.7 Combining Functions

We can define new functions by adding, subtracting, multiplying, or dividing other functions.

Let $f$ and $g$ be functions with domains $A$ and $B$ respectively. Then we define the functions $f + g$, $f - g$, $fg$ and $f/g$ as follows:

- $(f + g)(x) = f(x) + g(x)$ Domain: $A \cap B$
- $(f - g)(x) = f(x) - g(x)$ Domain: $A \cap B$
- $(fg)(x) = f(x)g(x)$ Domain: $A \cap B$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ Domain: $\{x \in A \cap B \mid g(x) \neq 0\}$

Examples:

(2.7 # 6) Let $f(x) = \frac{2}{x + 1}$ and $g(x) = \frac{x}{x + 1}$. Find $f + g$, $fg$, $f/g$ and their domains. Then find $(f + g)(2)$ and $(fg)(1)$.

Let $f(x) = \sqrt{8 + x}$ and $g(x) = \sqrt{x^2 - 4}$. Find $f - g$, $f/g$ and their domains. Then find $(f - g)(3)$ and $(f/g)(3)$. 
We can also combine functions by finding the **composition** of two functions.

If \( f \) and \( g \) are two functions, the composition of \( f \) and \( g \), denoted \( f \circ g \) is defined by
\[
(f \circ g)(x) = f(g(x))
\]
In other words, we evaluate \( g \) first, and then we plug in whatever we get into \( f \).

The domain of \( f \circ g \) consists of the values of \( x \) where both \( g(x) \) and \( f(g(x)) \) are defined.

**Examples**

- For \( f(x) = x + 2 \) and \( g(x) = x^2 \), find \( f(g(2)) \) and \( g(f(2)) \).

- For \( f(x) = \frac{x^3}{4} \) and \( g(x) = 2x - 3 \), find \( f \circ g \), \( g \circ f \), \( f \circ f \), and \( g \circ g \) and their domains.

From this example above, we see that \( f \circ g \) and \( g \circ f \) are not always the same function.

- For \( f(x) = \frac{1}{x^2 - 4} \) and \( g(x) = \sqrt{x + 4} \), find \( f \circ g \) and its domain.
• (2.7 #40) For $f(x) = \frac{2}{x}$ and $g(x) = \frac{x}{x + 2}$ find $f \circ g$, $g \circ f$ and their domains.

• For $f(x) = \frac{1}{x}$, $g(x) = x^2 - 1$, and $h(x) = x^3 + 2$, find $f \circ g \circ h$.

• (2.7 # 50) For $F(x) = \sqrt{1 + \sqrt{x}}$, find functions $f$ and $g$ so that $F(x) = f \circ g$.

2.8 One-to-One Functions and Their Inverses

A function $f$ is one-to-one if no two $x$’s in the domain have the same range value $f(x)$. In other words, whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

EVERY $x$-value must have a different $f(x)$ value.
There is an easy way to test if a function is one-to-one.

**Horizontal Line Test:** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Is \( f(x) = x^3 + 2 \) a one-to-one function?

Is \( f(x) = -|x| \) a one-to-one function?

Only one-to-one functions can have inverses.

If \( f \) is a one-to-one function with domain \( A \) and range \( B \), then the inverse function \( f^{-1} \) has domain \( B \) and range \( A \) and is defined by:

\[
f^{-1}(y) = x \text{ where } f(x) = y \text{ for any } y \text{ in } B.
\]

In other words, the inverse function \( f^{-1} \) “undoes” \( f \).

Domain of \( f^{-1} = \text{Range of } f \)

Range of \( f^{-1} = \text{Domain of } f \)

Note: \( f^{-1} \) does NOT mean \( \frac{1}{f} \). It is just notation.

Example: If \( f(1) = 3 \) and \( f(7) = -1 \), what are \( f^{-1}(3) \) and \( f^{-1}(-1) \)?
Inverse Function Property: If $f$ has inverse function $f^{-1}$, then $f^{-1}(f(x)) = x$ (for all $x$ in the domain of $f$) and $f(f^{-1}(x)) = x$ (for all $x$ in the domain of $f^{-1}$).

Any function $f^{-1}$ satisfying these properties is the inverse of $f$.

Show that the following functions are inverses of each other:

$$f(x) = \frac{3 - x^3}{4}, \ g(x) = (3 - 4x)^{1/3}$$

To find the inverse of a function, write $y = f(x)$. Solve for $x$ in terms of $y$ and then interchange $x$ and $y$. The resulting equation is $y = f^{-1}(x)$. (If you’d rather interchange $x$ and $y$ first and then solve for $y$, that is also fine.)

Examples: (2.8 #s 44 and 46) Find the inverses for the following functions.

$f(x) = (2 - x^3)^5$

$$f(x) = \sqrt{2x - 1}$$
\begin{align*}
(2.8 \ #38) \ f(x) &= \frac{x - 2}{x + 2} \\
\end{align*}

Graphically, the inverse function $f^{-1}$ is a reflection of $f$ across the line $y = x$.

Why? If $(a, b)$ is a point on $f$, then $(b, a)$ is a point on $f^{-1}$.

Example: (2.8 \ # 54) Let $f(x) = x^3 - 1$. Graph $f^{-1}$. 