Chapter 2: Functions

2.1 What is a Function?

Functions are often used in real-world situations when one thing depends on another thing.

- The area of a circle is a function of the radius.
- The average temperature of a city is a function of its latitude.
- Your grade in a class is a function of the amount of time you study.

A function $f$ is a rule that assigns to each element $x$ in its domain exactly one element, called $f(x)$, in its range.

If we have that $y = f(x)$, we say $y$ is a function of $x$, and call $x$ the independent variable and $y$ the dependent variable.

The domain of $f$ is the set of values $x$ can be.

The range of $f$ is the set of all possible values of $f(x)$ (the $y$ values).

We have already talked about domain. Remember that the denominator cannot be 0 and that anything under a square root (or any even root) must be greater than or equal to 0.

To find range, the easiest way is to look at the graph of the function.

Example: What are the domain and range of $y = |x|$?

Find the domain of the function $f(x) = \frac{\sqrt{25 - x^2}}{x - 2}$.
If we have a function \( y = f(x) \) and want to find \( f(a) \), we just substitute \( a \) wherever there is an \( x \) in the function.

Example: For \( f(x) = x^2 + x + 7 \), calculate \( f(-2), f(a), f(a + h), \) and \( \frac{f(a+h)-f(a)}{h} \).

The last part above is called the **difference quotient** and will be used in calculus in the definition of the derivative.

A **piecewise function** is a function which is defined differently on separate parts of its domain.

Example: Find \( f(-5), f(-1), f(1), f(4), f(6) \) for the following piecewise function.

\[
 f(x) = \begin{cases} 
 3x & \text{if } x \leq -1 \\
 1 & \text{if } -1 < x \leq 4 \\
 (x - 2)^2 & \text{if } x > 4 
\end{cases}
\]

### 2.2 Graphs of Functions

You need to be familiar with the graphs of the functions (except the greatest integer function) in the table on p. 166 of your textbook. These are the graphs we will be working with a lot.

We can graph functions by making a table of values and then sketching the graph. This is not always a perfect method, though, and it is sometimes necessary to use a graphing calculator.

Sketch a graph of the following functions and find the domain and range.

\[
 f(x) = \begin{cases} 
 x^2 - 1 & \text{if } x < 0 \\
 1 & \text{if } 0 \leq x \leq 2 \\
 x - 1 & \text{if } x > 2 
\end{cases}
\]
\[ f(x) = x - |x| \]

Use your graphing calculator to find the range of the function \( f(x) = -x^4 + 3x^3 - 4x \). Round any decimals to 4 places.

How can you tell if a graph is that of a function?
**Vertical Line Test:** A graph in the coordinate plane is the graph of a function if and only if there is no vertical line that intersects the curve more than once.

(In other words, if any vertical line crosses a graph more than once, it is not a function.)

How to tell if an equation defines a function: If you can solve for \( y \) in the equation and only get one value of \( y \) for any given value of \( x \), then the graph is a function.

Examples: Determine whether the equation defines \( y \) as a function of \( x \).

\[
x^2 + y = 9 \quad x^2 + (y - 1)^2 = 4 \quad x - 2|y| = 0
\]
2.3 Increasing and Decreasing Functions; Average Rate of Change

A function \( f \) is **increasing** on an interval if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \) in that interval.

A function \( f \) is **decreasing** on an interval if \( f(x_1) > f(x_2) \) whenever \( x_1 < x_2 \) in that interval.

In other words, look at the function from *left to right*. If its going up, its increasing, if its going down, its decreasing.

Determine graphically on what intervals the function \( f(x) = x^3 - 5x^2 + 4x + 1 \) is increasing and decreasing.

The **Average Rate of Change** of a function \( y = f(x) \) between \( x = a \) and \( x = b \) is

The average rate of change is the slope of the secant line between \( x = a \) and \( x = b \).

Find the average rate of change for the following functions on the given interval.

- \( f(x) = 5x^2 + x \) from \( x = -2 \) to \( x = 0 \), and then from \( x = a \) to \( x = a + h \).
• \( f(x) = 3x + 1 \) from \( x = 2 \) to \( x = 4 \), and then from \( x = a \) to \( x = a + h \).

• \( f(x) = \frac{6}{x+2} \) from \( x = 0 \) to \( x = h \).

If I am given distance as a function of time, \( d(t) \), then the average rate of change of \( d \) over a time interval is the same as the average speed.

Example: Suppose the distance of an object from its starting point is related to time by the function \( d(t) = \sqrt{t^2 + t} \). Find the average speed of the object over the interval from \( t = 0 \) to \( t = 3 \).