Chapter 1: Fundamentals

1.1 Real Numbers

Types of Real Numbers:

- **Natural Numbers**: \{1, 2, 3, \ldots\}; These are the counting numbers.
- **Integers**: \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}; These are all the natural numbers, their negatives, and 0.
- **Rational Numbers**: \{\frac{m}{n} | n, m \text{ are integers}, n \neq 0\}; These are ratios of integers (fractions). All numbers with repeating decimals can be written as fractions.
- **Irrational Numbers**: These are the nonrepeating, nonterminating numbers. Ex: \pi, \sqrt{2}, \sqrt[3]{4}, \ldots
- **Real Numbers**: All of the above put together.

Example: Consider the list of numbers 12, 1.987, \(-\frac{5}{2}\), \frac{\pi}{4}, -4, -\sqrt{5}, 0, 2.57. List the numbers that are:

- Natural numbers
- Integers
- Rational numbers
- Irrational numbers

Real numbers can be represented by points on the **real line**, or **number line**.

If a number \(a\) is further to the left than \(b\) on the number line, then \(a < b\). If \(a\) is to the right of \(b\), then \(a > b\).

**Properties of Real Numbers**

- **Commutative Property**: \(a + b = b + a\), \(ab = ba\)
- **Associative Property**: \(a + (b + c) = (a + b) + c\), \((ab)c = (ab)c\)
- **Distributive Property**: \(a(b + c) = ab + ac\), \((a + b)c = ac + bc\)

Example: Write \(-8x(3 - 2y)\) without parentheses.
Some Properties of Fractions

1. \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \)

2. \( \frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \) (Multiply by the reciprocal.)

3. \( \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \)
   Note: You can only do this when the fractions have the same denominator: \( \frac{a}{b} + \frac{a}{c} \neq \frac{a}{b+c} \)

4. If \( \frac{a}{b} = \frac{c}{d} \), then \( ad = bc \) (Cross multiplying.)

5. \( \frac{ca}{cb} = \frac{a}{b} \) (Cancel common factors.)

To add fractions with different denominators, we have to use the Least Common Denominator.

Example: \( \frac{5}{12} + \frac{7}{45} \)

A set is a collection of objects, which are called elements.

Sets of real numbers that make up portions of the number line can be represented by intervals.

A closed interval, \([a, b] = \{ x \mid a \leq x \leq b \} \); the set of all number between \( a \) and \( b \), including the endpoints.

An open interval, \((a, b) = \{ x \mid a < x < b \} \); the set of all numbers between \( a \) and \( b \), not including the endpoints.

An interval can also be half-open, half-closed: such as \((a, b] \) or \([a, b) \).

Parentheses indicate that the endpoint IS NOT included. Square brackets indicate that the endpoint IS included.
The union of two sets or intervals $A$ and $B$, denoted $A \cup B$, is the set of elements that:

The intersection of two sets or intervals $A$ and $B$, denoted by $A \cap B$, is the set of elements that:

Example: Graph the set $(-\infty, 7] \cup [2, 8)$.

Example: Graph the set $[-4, 5] \cap (1, \infty)$

The absolute value of a number $x$, denoted $|x|$, is how far $x$ is away from the origin. It is a measure of distance, so it must always be greater than or equal to 0.

$|x| =$

Example: $|2 - | -12|| = |\pi - 4| =$

The distance between two points $a$ and $b$ is: $d(a, b) = |b - a|$ or equivalently $|a - b|$.

Example: Find the distance between the numbers $-4$ and $-\frac{19}{4}$. 
1.2 Exponents and Radicals

Exponent Rules

1. \( a^n = a \cdot a \cdot a \cdot \ldots \cdot a \)
2. \( a^0 = 1 \), for \( a \neq 0 \)
3. \( a^{-n} = \frac{1}{a^n} \), for \( a \neq 0 \)
4. \( a^m a^n = a^{m+n} \)
5. \( \frac{a^m}{a^n} = a^{m-n} \)
6. \((a^m)^n = a^{mn}\)
7. \((ab)^n = a^n b^n \) Note: This only works with multiplication. \((a + b)^n \neq a^n + b^n\)
8. \((\frac{a}{b})^n = \frac{a^n}{b^n}\)
9. \((\frac{a}{b})^{-n} = (\frac{b}{a})^n\)
10. \(\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}\)

Examples: Simplify and write without negative exponents.

- \((4x^4y)^2 \left( \frac{x}{3y^2} \right)^3\)

- \(\left( \frac{xy^{-2}z^{-3}}{x^2y^3z^{-4}} \right)^{-2}\)
Radicals

The symbol √ means “the positive square root of”.

To find √a, find the number b such that b² = a. In all cases (except when a = 0), there are two numbers b that make this true. The answer is the POSITIVE one.

Similarly, to find ∛a, the cube root (or third root) of a, find the number b such that a = b³.

In general, to find √ⁿᵃⁿ, find the number b such that bⁿ = a.

- If n is EVEN, we must have a ≥ 0 and b ≥ 0.
- If n is odd, there are no restrictions.

Does √ⁿᵃⁿ always equal a?

Does ∛ⁿᵃⁿ always equal a?

So, we define √ⁿᵃⁿ =

Simplify the following expressions:

- √⁻ˡᵃ⁽ˣ⁺ʸ⁻¹⁰⁻¹²⁾⁻⁴⁺ʸ⁺¹⁰⁺⁻²⁵
Properties of Radicals

1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
2. $\sqrt[n]{a} = \sqrt[n]{a}$
3. $\sqrt[n]{b} = \sqrt[n]{b}$

Examples: Simplify these expressions

1. $\sqrt[2]{25}$

2. $\sqrt{6} \sqrt{15}$

3. $\sqrt[3]{27} \sqrt[64]{4}$

4. $\sqrt{32} + \sqrt{128}$

Note: It is NOT true that $\sqrt{a^2 + b^2} = a + b$.

All these properties follow from properties of exponents when we use the following:

$\sqrt[n]{a} = a^{1/n}$

$\sqrt[n]{a^m} = a^{m/n} = (\sqrt[n]{a})^m$

This means that ANY radical can be written using rational exponents.

Examples:

1. Calculate $36^{3/2}$

2. Calculate $48^{-3/4}$
3. Simplify the following and write without negative exponents. Assume all variables represent positive
numbers.
\[
\left( \frac{8x^{-5/2}y^6}{54z^3} \right)^{-2/3}
\]

4. Write the following as a single power of \( x \):
\[
\frac{\sqrt{x} \cdot \sqrt[3]{x}}{(\sqrt[3]{x})^2}
\]

We rationalize the denominator in order to get any radicals out of the denominator. We do this by
multiplying top and bottom by an appropriate factor.

Examples

- \( \frac{2}{\sqrt{5}} \)

- \( \frac{3}{\sqrt[3]{x^3}} \)