3.1 Graphing Systems of Linear Inequalities

An inequality is an expression like $x + 3y < 4$ or $2x - 5y \geq -2$. They look just like equations except the equal sign is replaced by one of $<, >, \leq, \geq$.

An inequality does not have just one solution. There will be many points $(x, y)$ that satisfy an inequality.

Steps to graphing an inequality.

1. Replace the inequality sign by an equal sign and graph the line. If the inequality is $<$ or $>$, use a dashed line. If the inequality is $\leq$ or $\geq$, use a solid line.

2. Pick a test point in one of the half planes formed by the line. Plug in the point into the inequality and determine if it makes the inequality true or false. It is usually easiest to pick $(0,0)$ unless it is actually a point on the line.

3. If the test point makes the inequality true, then that side of the line is the solution set $S$ and we shade it. If the test point makes the inequality false, the other side of the line is the solution set and we shade it. Always label the solution set $S$.

Examples:

$y \geq 3$

$-2x + y > 0$

$5x - 3y \geq 15$
If we have a system of linear inequalities, the solution set $S$ to this system is the set of all points that satisfy EVERY inequality in the system. The **corner points** of a solution set $S$ are the points where the boundary line of the region changes.

A solution set is **bounded** if it can be enclosed by a circle. Otherwise, it is **unbounded**. In the examples above, are the solution sets bounded or unbounded?

Examples: Graph these systems of inequalities and find the coordinates of any **corner points** of the solution set.

- $x + y \geq -2$
  $3x - y \leq 6$

- $x + y \geq 2$
  $2x + y \leq 6$
  $2x - y \geq -1$
  $x \geq 1$
  $y \geq 0$
3.2 Linear Programming Problems

The idea of linear programming problems is that we are given something that we want to maximize or minimize subject to some constraints.

Linear programming problems consist of an **objective function** (the function we are maximizing or minimizing) and **constraints** in the form of linear equations or inequalities. The solution set $S$ to this system of constraints is called the **feasible region**.

Examples: Set up the following linear programming problems.

- (Section 3.2 #8, Tan) A farmer plans to plant two crops, A and B. The cost of cultivating crop A is $40/acre, whereas the cost of crop B is $60/acre. The farmer has a maximum of $7400 available for land cultivation. Each acre of crop A requires 20 labor-hours and each acre of crop B requires 25 labor-hours. The farmer has a maximum of 3300 labor-hours available. If she expects to make a profit of $150/acre on crop A and $200/acre on crop B, how many acres of each crop should she plant in order to maximize her profit?

- (modified from *Finite Mathematics* by Long & Graening) Suppose in a diet, you are only allowed to eat two types of sandwiches. Sandwich A has 5 grams of carbs, 80 calories, and 2 grams of fat. Sandwich B has 7 grams of carbs, 110 calories, and 5 grams of fat. Each week, you need to eat at least 400 calories but no more than 105 grams of carbs. Also, you are required to eat at least 3 times as many Sandwich A’s as Sandwich B’s. How many of each type of sandwich should you eat in a week in order to minimize your fat intake.
• (modified from Section 3.2 #16, Tan) Ashley has earmarked at most $250,000 for investment in three mutual funds: a money market fund, an international equity fund, and a growth-and-income fund. The money market fund has a rate of return of 6%/year, the international equity fund as a rate of return of 10%/year, and the growth-and-income fund has a rate of return of 15%/year. Ashley has stipulated that the amount invested in the money market fund should be no more than three times the amount invested in the international equity fund. Additionally, Ashley stipulates that at least $10,000 should be invested in the growth-and-income fund. How much should Ashley invest in each type of fund to maximize the return on her investment?

3.3 Graphical Solutions of Linear Programming Problems

Now that we’ve set them up, how do we find the solution to the linear programming problem?

If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible region $S$ associated with the problem.

Suppose we are given an objective function $P = ax + by$ and a feasible region $S$.

• If $S$ is BOUNDED, then $P$ WILL have both a maximum and a minimum value.

• If $S$ is UNBOUNDED, both $a$ and $b$ are $\geq 0$, and the constraints include the inequalities $x \geq 0$ and $y \geq 0$, then $P$ will have a minimum value, but no maximum value.

To solve linear programming problems, we use the Method of Corners:

1. Graph the feasible region. Label it $S$.
2. Find the coordinates of all corner points.
3. Evaluate the objective function at each corner point.
4. Find the corner point(s) that gives the maximum or minimum value (if one exists).
   (a) If there is only one vertex where the max/min is attained, this is a unique solution.
   (b) If the objective function is optimized at two adjacent vertices of $S$, then it is optimized at EVERY point along the line segment connecting these two vertices. In this case, there are infinitely many solutions.
Examples:

• (Modified from Section 3.3, #15, Tan)
  Minimize $C = 2x + 3y$
  Subject to:
  $x + y \geq 3$
  $x + 2y \geq 4$
  $x \geq 0$
  $y \geq 0$

If I wanted to maximize $C$ subject to the same constraints above, where would this occur?
• (Section 3.3, #9, Tan)
  Maximize $P = 2x + y$
  Subject to:
  $x + y \leq 4$
  $2x + y \leq 5$
  $x \geq 0$
  $y \geq 0$
• Maximize AND Minimize $C = 10x + 12y$

Subject to:
$19x + 3y \geq 95$
$x + y \geq 18$
$16x + 9y \leq 216$
$x \geq 0$
$y \geq 0$
Suppose a company makes toy cars out of paint and wood. A standard toy car requires 2 cans of paint and 4 blocks of wood. A collector’s edition toy car requires 2 can of paint and 2 blocks of wood. The company has at most 16 cans of paint available and at most 20 blocks of wood available each day. Each standard toy car brings in a profit of $10 and each collector’s edition toy car brings in a profit of $16. The company stipulates that the number of standard cars made should be at most twice the number of collector’s edition cars made. Further, they will make no more than 7 collector’s edition cars a day. How many toy cars of each type should the company make in order to maximize their profit?

How much of each resource (paint and wood) is left over at the optimal level of production?