Challenge Seminar Notes Volume 1

At our Sept 4 meeting, we discussed these problems:

Let $C(\alpha)$ denote the coefficient of x^{1992} in the power series expansion of $(1+x)^\alpha$. Evaluate

$$\int_{y=0}^{1} C(-y-1) \sum_{k=1}^{1992} \frac{1}{k+y}\, dy$$

There were two hurdles. The first of these was a matter of memory and accuracy: just what IS the formula for $C(\alpha)$? Here, the approach taken was to fix numerical values for α such as 3 or 1/2 and evaluate the first few derivatives of $(1+x)^\alpha$ and then set $x = 0$. This jogs the memory and one recalls that the binomial theorem holds also for non-integer exponents.

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

where $\binom{\alpha}{k} := (\alpha-0)(\alpha-1)\ldots(\alpha-(k-1))/k!$. Thus $C(-y-1) = (y+1)(y+2)\cdots(y+1992)$ and the integrand in the question is a sum of polynomials in y.

Rather than write it out and multiply it out, a horrible task and quite impossible within the alloted time, the thing to do is to back off to small cases. How would one look at $\int((y+2)(y+3)+(y+1)(y+3)+(y+1)(y+2))\, dy$ for example? This can be done without too much trouble by pencil and paper, and likewise for the cases $k = 2$ or $k = 4$. Then you have to notice something and then everything falls into place.

We next considered this problem: Suppose $S = \{a_1, a_2, \ldots a_n\}$ is a set of n distinct real numbers, (which we take to be presented in increasing order.) Let A be the set of all averages of any two of these numbers. What is the MINIMUM number of elements in A? The approach taken was to consider the special case of $n = 5$ and 'wing it' in an attempt to guess the answer. The best way to hold down the number of different averages seemed to be to use $S = \{1, 2, 3, 4, 5\}$ or an equivalent list, and then we thought about two issues: How many elements would A have, as a function of n, if you used this approach, and secondly, why is it not possible to do any better using some other kind of set S?

We finally considered one of the two cases, the case $a > 1$, in the following problem: Find

$$\lim_{x\to\infty} \left[\frac{1}{x}\left(\frac{a^x-1}{a-1}\right)\right]^{1/x}$$

The other case was $0 < a < 1$. For the case $a > 1$ we used the L'Hospital's rule and logarithms and argued that $a^x - 1$ was close enough to a^x that taking the $1/x$ power would make the answer be a. The difficulty lay in seeing how to express this basic idea with sufficient accuracy and rigor.