

Problem B1 2001

Let n be an even positive integer. Write the numbers $1, 2, \dots, n^2$ in the squares of an $n \times n$ grid so that the k th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \dots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

OK there it is. The first difficulty with a problem like this is to read it. After due deliberation, one constructs an example that is small enough to be tractable, yet large enough to have a chance of being typical. Something like this:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Next, one has to cast about for some reason, beyond the mere chance of arithmetic, why it should have happened that both color sums came to the same value. This casting about is the hard part, or one of the hard parts, so you cannot expect to just see it at a glance unless you're having a very good day. Look on the back of this sheet for a hint.

We now turn to another question, this one from an earlier contest.

Problem A2 2000 Prove that there exist infinitely many integers n such that n , $n+1$ and $n+2$ are each the sum of two squares of integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

What is one to do? Well, if there are infinitely many, perhaps you can find another example? A diligent search will disclose that $8 = 4 + 4$, $9 = 9 + 0$, and $10 = 9 + 1$, and that $288 = 12^2 + 12^2$, $289 = 17^2 + 0^2$, and $290 = 13^2 + 11^2$. Now it helps to know some things about recurrence relations and the square root of 2 and so forth. Try this: Use Newton's method to look for a solution of $x^2 - 2 = 0$ starting with $x = 1$. You should get $3/2$ and then $17/12$. These numbers are numbers that have showed up in the search. A *clue!* Now, can you formulate a statement which captures what has been going on in these examples? (If you had a statement, you'd have a better chance of proving it, than if you didn't.)

Here is the promised hint for the first problem.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

=

1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4

+

0	0	0	0
4	4	4	4
8	8	8	8
12	12	12	12