

Quick and dirty notes: We talked about volumes of rotation this Sept. 4.

Cavalieri principle from geometry: shapes that have the same cross sectional areas all the way up and down, have the same volume. Thus, the cylinder with base $x^2 + y^2 \leq R^2$ and height H has the same volume as a rectangular solid with base $0 \leq x \leq R$, $0 \leq y \leq \pi R$, and height H , it is $V = \pi R^2 H$.

We could use this to get the volume of a cone with a circular base of radius R and a height of H , and comparing it to a pyramid with rectangular base R by πR and height H , where the volume of the pyramid can be got by geometry of straight-line edged solids; it's base times height over 3. This would give $\pi R^2 H/3$ for the cone. Our calculus method is to integrate $\pi r^2 ds$, the circular disks idea.

With the special case $r = s$, got by rotating the triangle with corners at $(0, 0)$, $(0, 1)$, and $(1, 1)$ about the y axis, we got a volume of $\int_0^1 \pi y^2 dy = \pi/3$.

For the unit sphere, we got $\int_{-1}^1 \pi(1-s^2) ds = 4\pi/3$ and for a sphere of radius R we got $4\pi R^3/3$.

Then we had some fun with the idea, going into four dimensions to compute the volume of the four-dimensional sphere. A point (x, y, z, w) sits inside a unit 4-D sphere if and only if $x^2 + y^2 + z^2 + w^2 \leq 1$, and the cross section at $s = w$ is the 3-D sphere $x^2 + y^2 + z^2 \leq 1 - w^2$. That's a sphere of radius $\sqrt{1 - w^2}$, and it has volume $(4/3)\pi(1 - w^2)^{3/2}$. Integrate

$$\int_{w=-1}^1 \frac{4}{3}\pi(1-w^2)^{3/2} dw = \frac{\pi^2}{2}$$

to get the 4-D volume.

As a check, since we cannot very well construct a 4-D sphere and fill it with ultra-heavy water, (!) and emulate Archimedes, we shall have to fill our sphere with random points.

We pick a million points, lists of four random numbers, each taken from $[-1, 1]$, and evaluate for each, whether or not it falls inside the 4-sphere, that is, whether or not $w^2 + x^2 + y^2 + z^2 \leq 1$. The 4-cube of side 2 in each of its four dimensions has 4-D volume of 16, so the probability that a random point sits inside our sphere ought to be $\pi^2/32$.

Rotate the region bounded by $y = x^2$, the y axis, and the horizontal line $y = 1$, about the y axis. What will be the volume?

Rotate it about the x axis. Now you need washers, as it's a trumpet mouth-piece with zero aperture.

Take the region bounded by $y = x^2$ and $y = x$ about the x axis. Again you need washers. $\int_0^1 \pi(x^2 - x^4) dx$.

Rotate this region about the line $y = x$. This is much harder. You need to reframe the algebra in terms of new coordinates s and r , where s runs along the line $y = x$ and r runs perpendicular. The details are found in the Maple sheet posted online.

Rotate the region bounded by the line $y = 3$, the y axis, and the curve $y = x^3 + x^2 + 1$ about the x axis. Easy: washers, $\int_0^1 \pi(3^2 - (x^3 + x^2 + 1)^2) dx$ and then some drudgery. Rotate it about the y axis? This gets you into a technical difficulty. The head on approach would be to solve for x in terms of

y , say $x = f(y)$ whenever $y = x^3 + x^2 + 1$ and then integrate $\int_{y=1}^3 \pi f(y)^2 dy$. But that's a stinker of an algebra task, and when done, if we did somehow do it (in the Renaissance, in Italy, they did figure out how to solve a general cubic, and we could look it up and do it!) the resulting calculus would figure to be a nightmare.

However! The substitution $\theta = f(y)$, or equivalently $y = f^{-1}(\theta) = \theta^3 + \theta^2 + 1$, and $dy = 3\theta^2 + 2\theta$, gives

$$\pi \int_{\theta=0}^1 \theta^2(3\theta^2 + 2\theta) d\theta = \frac{11\pi}{10}.$$

We get an answer with a somewhat mind bending substitution. But what's mind bending the first time is more natural the second, and pretty routine eventually. The inverse function is your friend.

Anyhow, there's an entirely different approach to the same question. Break the volume up as a Russian-doll stack of nested cylindrical shells. It's done on the Maple sheet posted to my 152H course home page.

The resulting integral is

$$\int_0^1 2\pi x h(x) dx = \int_0^1 2\pi x(3 - (1 + x^2 + x^3)) dx.$$

This, too, works out to $11\pi/10$, a happy miracle of mathematics. The same question has the same answer, whichever way we work it. Kind of like a card trick.

Two very different integrals evaluated to the exact same answer. Might there be a computational technique in there somewhere? Yes, and it's called integration by parts, and we'll come to it eventually by a very different logical path.

Now, we come to the problems we did in class. (You did, at the board). All of y'all should go over these and figure out either the answer, or an integral which if evaluated, if only the techniques were available, would give the answer. In each case, find the volume got by rotating the region specified about the specified axis.

1. $0 \leq y \leq x^2$, $0 \leq x \leq 1$, about the x axis.
2. $0 \leq y \leq x^2$, $0 \leq x \leq 1$, about the line $y = -1$.
3. $\sin x \leq y \leq 1$, $0 \leq x \leq \pi/2$, y axis.
4. $0 \leq y \leq x - x^2$, about the y axis.
5. Same region, but about the y axis.