

A calculus problem [hint hint] Prove that for every integer  $n > 1$ ,

$$\frac{3n+1}{2n+2} < \frac{1}{n^n} + \frac{2^n}{n^n} + \dots + \frac{n^n}{n^n} < 2$$

More of a hint: Think about the function  $x^n$ .

A discrete problem that uses Gaussian integers.

Here's the facts you need:

- (1) the Gaussian integers are the numbers  $a + bi$  where  $a$  and  $b$  are ordinary integers.
- (2) Gaussian integers have the same unique factorization structure that ordinary integers have.

With the ordinary integers, there are two units,  $\pm 1$ , and every number can be written uniquely in the form  $u \cdot \prod_1^r p_j^{e_j}$  where  $r$  is a positive integer,  $u$  is one of the units, the  $p_j$  are prime, and the  $e_j$  are positive integers, with  $2 \leq p_1 < p_2 \dots < p_r$ .

With the Gaussian integers, there are four units,  $\pm 1$  and  $\pm i$ . Every number can be written in the form  $u \cdot \prod_1^r p_j^{e_j}$  where the  $p_j$  are now Gaussian primes. Same conditions as before on  $r$  and the  $e_j$ .

The Gaussian primes are the Gaussian integers that are not zero and not a product of two non-units. For instance,  $1 + i$  is a Gaussian prime, as are  $1 \pm 2i$ . Now  $2 \pm i$  are also both prime, and  $(2 + i)(2 - i) = (1 + 2i)(1 - 2i) = 5$ . So one must be careful about claiming unique factorization here. However, we could have observed with ordinary integers that  $2 * 3 = (-2) * (-3) = 6$ , and nothing more dramatic is going on here:  $2 + i = (1 - 2i) * i$  so the two primes involved are not really different any more than were  $\pm 2$ . Factorization in the Gaussian integers is unique in the same way as for ordinary integers: primes which are unit multiples of each other don't count as different, and the order in which the factors are presented doesn't count, and apart from that, the factorization is unique.

Now here is the problem: Prove that for all positive (ordinary) integers  $p$  and  $q$ , if  $a > 2$  is an integer then

$$((1 + ai)/\sqrt{1 + a^2})^q \neq ((1 + 2i)/\sqrt{5})^p$$