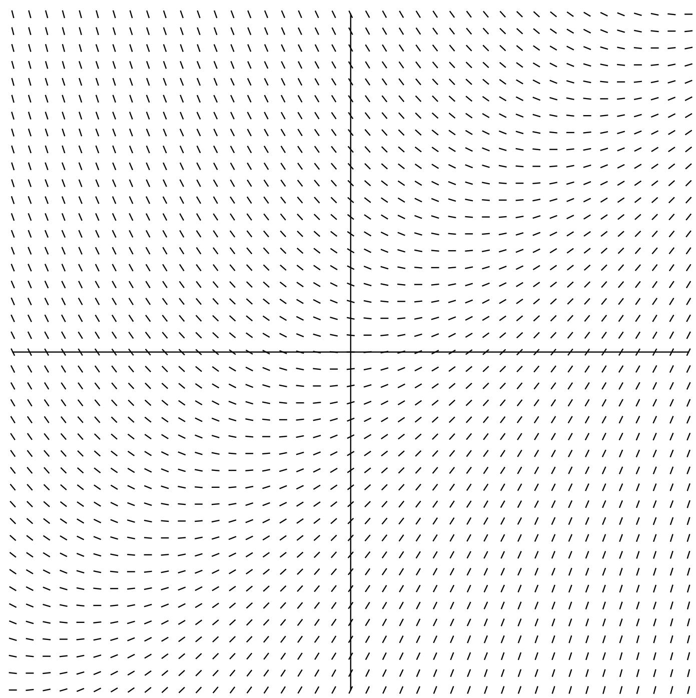
Exam 1 Math 308, Spring 2008

1. (a) Sketch a direction field, over the region $-2 \le x \le 2$, $-2 \le y \le 2$, for the differential equation $y' = x - y$.



   (b) Solve the initial value problem $y' = x - y$ with $y(0) = 0$. This is first order linear, and may be put in standard form as $y' + 1y = x$. The integrating factor is e^x, so the DE is equivalent to $y'e^x + ye^x = xe^x$ and thus to $(ye^x)' = xe^x$. Integrating by parts on the right and by stripping the prime off the left gives $ye^x = xe^x - 1e^x + C$ so that $y = x - 1 + Ce^{-x}$. Now we want $y(0) = 0$, so $C = 1$ and the answer is $y = x - 1 + e^{-x}$.

2. Solve the initial value problem $y' = xy^2$ with $y(0) = 1$. This is separable (it is not linear), and on separating variables we learn that $dy/y^2 = xdx$. Integrating gives $-1/y = x^2/2 + C$. We want $y(0) = 1$ so $-1 = C$. Thus

$$y = \frac{1}{1 - x^2/2} = \frac{2}{2 - x^2}.$$

3. Consider the differential equation $y' = x - y^2$ and the solution $y = \phi(x)$ through $(1, 2)$.

(a) Use Euler's method with step size 0.1 to estimate $\phi(1.2)$. At $(1,2)$, the derivative is $1-2^2=-3$. Thus with $h=0.1$, $(x_1,y_1)=(1.1,1.7)$. Now the new derivative through *this* point is $1.1-1.7^2=-1.79$. With step size still 0.1, $(x_2,y_2)=(1.2,1.7-0.179)=(1.2,1.521)$.

(b) Find $\phi''(1)$. Hint: Differentiate the equation $y'=x-y^2$, keeping in mind that y is a function of x. So

$$y''=x'-(y^2)'=1-2yy'=1-2y(x-y^2)=1-2xy+2y^3.$$

Taking $x=1$ and $y=2$ gives $\phi''(1)=13$.

(c) Use this to get another estimate for $\phi(1.2)$, based on the idea of one step along a parabolic approximation to the solution. From the previous part, $\phi(1.2)\approx 2-3*(0.2)+(1/2!)*13*(0.2)^2=1.66$.

4. Solve $y''+y=0$ with initial conditions $y(0)=1$, $y'(0)=2$. $y=\cos(x)+2\sin(x)$. (Auxilliary equation was $r^2+1=0$, so $r=\pm i$ and that gives $y_1=\cos x, y_2=\sin x$. Now, fit the initial conditions.)

5. Solve $y''+2y'+2y=x^2$ (general solution). $y=C_1e^{-x}\cos(x)+C_2e^{-x}\sin x$ gives the homogeneous part of the answer, because the auxilliary equation is $r^2+2r+2=0$ and the quadratic formula gives $r=1\pm i$. Now we need a *particular solution*, that is, a ϕ so that $L\phi=x^2$, where $Ly:=y''+2y'+2y$. The method of undetermined coefficients is nicely suited to this. We have $L1=2$, $Lx=2x+2$, and $Lx^2=2x^2+4x+2$. So if we input $a_0+a_1x+a_2x^2$ to L, out comes $2a_0+(2x+2)a_1+(2x^2+4x+2)a_2$ which needs to be equal to $0+0x+1x^2$. This forces $a_2=(1/2)$, and then $a_1=-1$, and finally $a_0=1/2$ so the overall answer is

$$y=\frac{1}{2}x^2-2x+\frac{1}{2}+C_1e^{-x}\cos x+C_2e^{-x}\sin x.$$

6. Solve

$$y''-y=\frac{1}{e^x+1}$$

using variation of parameters. The corresponding homogeneous differential equation has these two linearly independent solutions: e^{-x} and e^x. The formula for v_1 and v_2 goes like this: $v_1=\int(-y_2g/W)$, $v_2=\int y_1g/W$. The substitution $u=e^x$ may help with the required integrations.

The formulas give

$$v_1=\int\frac{-e^x}{2(1+e^x)}\,dx=-\frac{1}{2}\ln(1+e^x)$$

for y_1, on using the suggested substitution. For y_2 it's harder. Here goes:

$$y_2=\int\frac{+e^{-x}}{2(1+e^x)}\,dx=\frac{1}{2}\int\frac{e^x}{e^{2x}(1+e^x)}\,dx=\frac{1}{2}\int\frac{du}{u^2(1+u)}.$$

This last integral breaks apart using partial fractions as

$$\frac{1}{2}\int\left(\frac{1}{u^2}-\frac{1}{u}+\frac{1}{u+1}\right)du=\frac{1}{2}\left(-\frac{1}{u}-\ln(u)+\ln(u+1)\right).$$

Converting back to x gives

$$v_1=-(1/2)\ln(1+e^x),\quad v_2=(1/2)(-e^{-x}-x+\ln(1+e^x)).$$

Now plug this in to the formula $y=v_1y_1+v_2y_2$ to get

$$y_{\text{particular}}=\frac{1}{2}\left(-e^{-x}\ln(1+e^x)+(-1-xe^x+e^x\ln(1+e^x))\right).$$

Add the homogeneous solutions based on y_1 and y_2 for the final answer.

7. Use the trick for Cauchy-Euler type problems to convert

$$y''+\frac{3}{x}y'+\frac{2}{x^2}y=x$$

into a second order nonhomogeneous linear differential equation with constant coefficients. Just say what the new differential equation is; don't solve it. The trick is to say "let $Y(t)=y(e^t)$". Equivalently, make the substitution $x=e^t$. The equation becomes $Y''+2Y'+2Y=e^{3t}$. (This is easily solved by undetermined coefficients.) The reason we have a 2 in place of the old 3 is that

$$\frac{d^2Y}{dt^2}=\frac{d}{dt}e^ty'(e^t)=e^ty'(e^t)+e^{2t}y''(e^t)$$

so

$$\begin{aligned}Y''+2Y'+2Y&=\left(e^{2t}y''(e^t)+e^ty'(e^t)\right)\\&\quad+2\left(e^ty'(e^t)\right)+2y(e^t)\\&=e^{2t}\left(y''+3e^{-t}y'+2e^{-2t}y\right)\\&=e^{t2}(y''+(3/x)y'+(2/x^2)y)=x^2(x)=x^3=e^{3t}.\end{aligned}$$

8. The mysterious differential equation $y''+p(x)y'+q(x)y=0$ has secret p and q; the only thing known about p and q is that they are both continuous functions. Your spy reports that while he hasn't been able to worm out p and q, he has seen the solutions file and $x+1$ and x^2+2x+2 are both solutions. What do you conclude?

   Your spy hasn't had a look at the solutions files, or if he has, he isn't telling you what they said. If $x+1$ and x^2+2x+2 are both solutions, then so is the combination $-2(x+1)+(x^2+2x+2)=x^2$. But the existence and uniqueness theorem says that x^2 cannot be a solution to a homogeneous linear differential equation with continuous p and q, because it matches the solution $y=0$ at zero, both as to value and first derivative. No two solutions match like that.