

The Putnam Challenge: A math 489 Seminar Course

The Putnam challenge course is a seminar style problems course. There are many questions of mathematics which on their face are quite difficult, yet can be answered with relatively little expenditure of ink and paper, provided one has the right idea. These questions call on the same side of human ingenuity as do chess combinations. As in chess, combinations are far from the only path to success, and strategy matters enormously. But even the best strategy can seldom be put into effect without good tactics, and the same is true of mathematics. Hence the best students have need of well honed problem solving skills to complement native mathematical talent and thorough study of the fundamentals.

The problems seminar will meet once weekly, in the evening from 7:00 to some time after 8:00, with the day chosen to accommodate the schedules of the participants. We will examine a variety of problems, taken in most cases from previously published contest problems. These problems are in an altogether different category of difficulty from what one encounters in regular classwork. Tried and true approaches don't work. We will study how one might nonetheless unravel such a problem and get to an answer. There are 'meta-approaches' to difficult problems. These rules of thumb are not methods, as they require judgment, moral courage, stamina and perhaps some flair to put into effect, but they can frequently convert a problem that initially looks like a blank cliff face into something that can be solved.

We will also pay some attention to the issue of exposition. Once one has the basic riddle riddled, how does one communicate the answer in such a way as to leave no room for doubt as to the correctness and completeness of the solution?

Participants will be invited to sketch their own solutions, or often, tentative observations which may somehow lead to a solution. From time to time, participants may wish to present a problem for class discussion. That is, the participant will take temporary command of the seminar and lead the discussion on a certain problem; this of course will mean that the participant needs to prepare by making a thorough study of the problem to be presented, the various known solutions, and the techniques behind such solutions. Participants are also expected to take some of the sometimes sketchy solutions arrived at during seminar discussion and convert them into their own fully developed proofs. In written proofs one has no opportunity for further give and take with the presumptive reader, whose likely doubts and objections must therefore be anticipated and preempted. The class is in its nature an honors course, but for those wanting formal honors credit, these expectations convert to elements of the honors contract. For each credit hour honors credit taken, the participant will be expected to present one problem and to furnish three properly written, fully developed proofs based on the discussion of problems presented by the seminar leader or by other participants. Revision of submitted proofs to get them into top form is expected and initial flaws are forgivable. Evaluation will be based on participation and , where applicable, writeups and presentations.

Problem B1 2001

Let n be an even positive integer. Write the numbers $1, 2, \dots, n^2$ in the squares of an $n \times n$ grid so that the k th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \dots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

OK there it is. The first difficulty with a problem like this is to read it. After due deliberation, one constructs an example that is small enough to be tractable, yet large enough to have a chance of being typical. Something like this:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Next, one has to cast about for some reason, beyond the mere chance of arithmetic, why it should have happened that both color sums came to the same value. This casting about is the hard part, or one of the hard parts, so you cannot expect to just see it at a glance unless you're having a very good day. Look on the back of this sheet for a hint.

We now turn to another question, this one from an earlier contest.

Problem A2 2000 Prove that there exist infinitely many integers n such that $n, n+1$ and $n+2$ are each the sum of two squares of integers.

[Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

What is one to do? Well, if there are infinitely many, perhaps you can find another example? A diligent search will disclose that $8 = 4 + 4$, $9 = 9 + 0$, and

$10 = 9 + 1$, and that $288 = 12^2 + 12^2$, $289 = 17^2 + 0^2$, and $290 = 13^2 + 11^2$. Now it helps to know some things about recurrence relations and the square root of 2 and so forth. Try this: Use Newton's method to look for a solution of $x^2 - 2 = 0$ starting with $x = 1$. You should get $3/2$ and then $17/12$. These numbers are numbers that have showed up in the search. A *clue!* Now, can you formulate a statement which captures what has been going on in these examples? (If you had a statement, you'd have a better chance of proving it, than if you didn't.)

Here is the promised hint for the first problem.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

=

1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4

+

0	0	0	0
4	4	4	4
8	8	8	8
12	12	12	12