1. Let \( n \) be the number of baseball cards Paul started with. He gave his third friend \( n - \frac{1}{2} n - \frac{1}{3} n = 12 \) cards, so \( 6n - 3n - n = 72 \), or \( n = 36 \) cards.

2. The remainder is the same as when \( 3^{2013} \) is divided by 10. The sequence of numbers \( 3^n \mid 10 \) is \{3, 9, 7, 1, 3, 9 \ldots \}, so \( 2013^{2013} \mid 10 = 3^{2013} \mid 10 = 3 \).

3. Let \( x \) and \( y \) be the numbers. Then \( xy = 12 \), and \( \frac{x^3 - y^3}{(x - y)^3} = \frac{x^2 + xy + y^2}{x^2 - 2xy + y^2} = \frac{x^2 + 12 + y^2}{x^2 - 24 + y^2} = 19 \). Clearing the fractions yields \( 18x^2 + 18y^2 = 12 + 19(24) \), or \( x^2 + y^2 = 26 \). Adding \( 2xy = 24 \) to both sides yields \( (x + y)^2 = 50 \), or \( x + y = 5\sqrt{2} \).

4. Since \( \triangle ABC \) is an equilateral triangle, \( \triangle ACE \) is a 30-60-90 triangle. Further, \( m \angle BCF = 120^\circ \), so \( \triangle OCE \) and \( \triangle OCF \) are also 30-60-90 triangles which are congruent to \( \triangle ACE \). Therefore, \( OF = AE = 3\sqrt{3} \).

5. Extend segments \( TA \) and \( UM \) until they intersect at point \( X \). Since \( \angle AMX = \angle UAX \), \( \triangle AMX \sim \triangle UAX \), so \( \frac{MX}{AX} = \frac{AM}{UA} = \frac{5}{10} \), or \( AX = 2MX \). From right triangle \( AMX \), \( MX^2 + (2MX)^2 = 5^2 \), or \( MX = \sqrt{5} \) and \( AX = 2\sqrt{5} \). Since \( \frac{AX}{UX} = \frac{5}{10} \) as well, we have \( UX = 2AX = 4\sqrt{5} \). Therefore, \( UM = UX - MX = 3\sqrt{5} \).

6. Let \( r \) and \( s \) be the roots of \( p(x) \). Then \( r + s = \frac{1}{3} \) and \( rs = -\frac{13}{2} \). The sum and product of the roots of \( q(x) \) must be \( \frac{1}{r} + \frac{1}{s} = \frac{r + s}{rs} = \frac{1}{13} \) and \( \frac{1}{r} \cdot \frac{1}{s} = -\frac{2}{13} \). Therefore, \( q(x) = x^2 + \frac{1}{13} x - \frac{2}{13} = 13x^2 + x - 2 \).

7. Let \( z \) and \( w \) be the two numbers. Since \( z^2 = w \) and \( w^2 = z \), we must have \( z^4 = z \), or \( z(z - 1)(z^2 + z + 1) = 0 \). If \( z = 0 \), \( w = 0 \) and the numbers are not distinct. Similarly, if \( z = 1 \), \( w = 1 \) and again the numbers are not distinct. Therefore, we must have \( z^2 + z + 1 = 0 \), or, from the quadratic formula: \( z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2} \). But \( w^4 = w \) as well, so we also have \( w = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2} \). Then \( z - w = \sqrt{3}i \) or \(-\sqrt{3}i \), so \( |z - w| = \sqrt{3} \).

8. \( \tan(3\theta) = \cot(4\theta) = \tan \left( \frac{\pi}{2} - 4\theta \right) \). Thus, \( 3\theta = n\pi + \left( \frac{\pi}{2} - 4\theta \right) \), or \( \theta = \frac{2n\pi + \pi}{14} \). Therefore, the smallest positive solution is \( \theta = \frac{\pi}{14} \).

9. \( \tan(54^\circ)(\cos(54^\circ) + \cos(162^\circ)) = \sin(54^\circ) + \frac{\sin(54^\circ) \cos(162^\circ)}{\cos(54^\circ)} = \sin(54^\circ) + \frac{\sin(216^\circ) - \sin(108^\circ)}{2 \cos(54^\circ)} = \sin(54^\circ) - \frac{\sin(36^\circ)}{2 \cos(54^\circ)} - \frac{2 \sin(54^\circ) \cos(54^\circ)}{2 \cos(54^\circ)} = -\frac{1}{2} \).

10. Let \( a \) be the \( x \)-coordinate of the point of tangency. Then \( m = f'(a) = 3a^2 \), and the equation of the tangent line is \( y = a^3 + 3a^2(x - a) \). This line intersects \( y = x^3 \) when \( a^3 + 3a^2x - 3a^2 = x^3 \), or \( x^3 - 3a^2x + 2a^2 = 0 \). Using the fact that the line is tangent at \( x = a \), we can factor the left as \( (x - a)^2(x + 2a) = 0 \), so the line intersect \( f \) at \( x = -2a \). Therefore, the slope of the tangent line at the point is \( 3(-2a)^2 = 12a^2 = 4m \).
11. By letting \( u = 2x \), we have \[ \int_0^2 f(2x) \, dx = \frac{1}{2} \int_0^4 f(u) \, du = \frac{1}{2} (6 + 12) = 9. \]

12. \( g'(x) = e^{-x} f'(x) - e^{-x} f(x) \). \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 + 2 = 3x^2 + 2 \) and \( f(x) = x^3 + 2x + f(0) = x^3 + 2x + 1 \). Therefore, \( g'(3) = e^{-3}(29) - e^{-3}(34) = -5e^{-3} \).

13. \( f(x) = 1 + x - x^2 - x^3 + x^4 + x^5 - x^6 - x^7 + \cdots = (1 - x^2 + x^4 + x^6 + x^8 \cdots) + (x - x^3 + x^5 - x^7 + x^9 + \cdots) \). Since the two expressions are convergent geometric series, their sum is \( \frac{1}{1 + x^2} + \frac{x}{1 + x^2} = \frac{1 + x}{1 + x^2} \).

14. For \( x > 0 \), \( \sec(\arctan(x)) = \sqrt{\tan^2(\arctan(x)) + 1} = \sqrt{x^2 + 1} \), so \( x_2 = \sqrt{2} \), \( x_3 = \sqrt{3} \), \( \cdots \), \( x_{2013} = \sqrt{2013} \).

15. Using algebra and calculus, we can show that \( C_1 \) and \( C_2 \) intersect in the point \( Q \left( \frac{r^2}{2}, \frac{\sqrt{4 - r^2}}{2} \right) \) and the equation of the line through \( P \) and \( Q \) is \( y = \frac{\sqrt{4 - r^2} - 2}{r} x + r \). Therefore, the \( x \)-intercept is \( a = -\frac{r^2}{\sqrt{4 - r^2} - 2} = \sqrt{4 - r^2} + 2 \), so \( \lim_{r \to 0^+} a = 4 \).

Alternately, we can find the limit geometrically using the figure below, defining \( R \) to be the desired \( x \)-intercept:

\[ \angle PQS, \angle OQT, \text{ and } \angle SQR \text{ are all right angles (first 2 subtend diameters; the last is complementary to the first), } \angle OQS = \angle TQR. \text{ Further, using appropriate right triangles we see that } \angle OSQ = \angle TRQ = 90 - \angle OPQ, \text{ so } \triangle SOQ \sim \triangle RTQ. \text{ But } \triangle SOQ \text{ is isosceles; therefore } \triangle RTQ \text{ is also isosceles and } QT = TR. \text{ As } r \to 0^+, \text{ point } Q \text{ approaches the origin, so } QT \to OT = 2 \text{ and } TR \to 2. \text{ Therefore, } OR \to 4, \text{ so } a \to 4. \]

16. Define \( f(t) = \int_0^1 \frac{\ln(tx + 1)}{x^2 + 1} \, dx \). Then \( f(0) = 0 \), \( f(1) \) is our desired solution, and \( f'(t) = \int_0^1 \frac{x}{(tx + 1)(x^2 + 1)} \, dx \). Using partial fractions, we obtain \( f'(t) = \frac{1}{t^2 + 1} \int_0^1 \left( -\frac{t}{tx + 1} + \frac{x + t}{x^2 + 1} \right) \, dx \)

\[ = \frac{1}{t^2 + 1} \left( -\ln(tx + 1) + \frac{1}{2} \ln(x^2 + 1) + t \arctan x \right|_{x=0}^{x=1} = \frac{1}{t^2 + 1} \left( -\ln(t + 1) + \frac{1}{2} \ln 2 + \frac{\pi t}{4} \right) . \]

Integrate both sides from 0 to \( t \): \( f(t) - f(0) = \frac{1}{2} \ln 2 \arctan t + \frac{\pi}{8} \ln(t^2 + 1) - \int_0^t \ln(x+1) \, dx \). Since the last term on the right is \( f(t) \), we have \( f(t) = \frac{1}{4} \ln 2 \arctan t + \frac{\pi}{16} \ln(t^2 + 1) \). Therefore, \( f(1) = \frac{\pi \ln 2}{8} \).
17. Since \( f'\left(\frac{a}{x}\right) = \frac{x}{f(x)} \), we also have \( f'(x) = f'\left(\frac{a}{x}\right) = \frac{a}{f\left(\frac{a}{x}\right)} = \frac{a}{xf\left(\frac{a}{x}\right)} \). Differentiate both sides to yield \( f''(x) = -\frac{a}{x^2f\left(\frac{a}{x}\right)} + \frac{a^2f'\left(\frac{a}{x}\right)}{x^3\left(f\left(\frac{a}{x}\right)\right)^2} \). Use the first two equations to substitute for \( f\left(\frac{a}{x}\right) \) and \( f'\left(\frac{a}{x}\right) \) to yield \( f''(x) = -\frac{f'(x)}{x} + \frac{(f'(x))^2}{f(x)} \). Clear the fractions and the right hand side to obtain \( xf(x)f''(x) + f'(x)f(x) - x(f'(x))^2 = 0 \). Divide by \( (f(x))^2 \):
\[
\frac{xf(x)f''(x) + f'(x)f(x) - x(f'(x))^2}{(f(x))^2} = 0, \text{ or } \frac{f(x)(xf''(x) + f'(x)) - f'(x)xf'(x)}{(f(x))^2} = 0.
\] Since the left side is the derivative of \( \frac{xf'(x)}{f(x)} \), this expression must be a constant (call it \( d \)). Then \( \frac{f'(x)}{f(x)} = \frac{d}{x} \). Integrate both sides to obtain \( \ln(f(x)) = d \ln x + C \), or \( f(x) = cx^d \). \( f(1) = 2 \) implies that \( c = 2 \), and \( f'(1) = cd(1)^{d-1} = 6 \) implies \( d = 3 \). Therefore, \( f(x) = 2x^3 \).

18. Let \( x = \log_c a \) and \( y = \log_c b \). Then \( 2 \left(\frac{1}{x} - \frac{1}{y}\right) = \frac{3}{x+y} \). Clearing the fractions yields \( 2y(x+y) - 2x(x+y) = 3xy \), \( 2y^2 - 3xy - 2x^2 = 0 \). \( 2y + x)(y - 2x) = 0 \). Therefore, we have \( 2 \log_c b = -\log_c a \) (not possible since \( a > 1 \) and \( b > 1 \)) or \( \log_c b = 2 \log_c a \). Therefore, \( \log_c b = \frac{\log_c b}{\log_c a} = 2 \).

19. If \( k \) is the location of the maximum value of \( f \), then we must have \( f(k) \geq f(k+1) \) and \( f(k) \geq f(k-1) \), or (since \( f(k) > 0 \) for all \( k \)) \( \frac{f(k)}{f(k+1)} \geq 1 \) and \( \frac{f(k)}{f(k-1)} \geq 1 \). The first inequality yields
\[
\binom{2013}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{2013-k} \geq 1, \text{ which simplifies to } \frac{5(k+1)}{2013-k} \geq 1 \text{ which is true when } k \geq \frac{2008}{6} = 334.66666666666667.
\]
In like manner, the second inequality simplifies to \( \frac{2014-k}{5k} \geq 1 \), which is true when \( k \leq \frac{2014}{6} = 335.66666666666667 \). Therefore, \( f \) is maximized when \( k = 335 \).