1. At a party, each man danced with exactly three women and each woman danced with exactly two men. Twelve men attended the party. How many women attended the party?

Solution Since each man danced with exactly three women and there were 12 men, then the total number of dancing pairs was $12 \cdot 3 = 36$. Since each woman danced with two men, then the number of women was $36/2 = 18$. The answer is 18.

2. Find the largest integer $n$ such that $2^n$ divides $2012!$, where $2012! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2012$.

Solution

Obviously,

$$n = \#(\text{of even numbers not greater than 2012}) + \#(\text{of numbers divisible by 4 and not greater than 2012}) + \#(\text{of numbers divisible by 8 and not greater than 2012}) + \ldots + \#(\text{of numbers divisible by 1024 and not greater than 2012})$$

(note that 1024 is the maximal power of 2 not greater than 2012).

Let $[x]$ be the integral part of $x$, i.e. the greatest integer less than or equal to $x$. Then the number of numbers divisible by $m$ and not greater than 2012 is equal to $\left\lfloor \frac{2012}{m} \right\rfloor$. Therefore from (1) it follows that

$$n = \left\lfloor \frac{2012}{2} \right\rfloor + \left\lfloor \frac{2012}{4} \right\rfloor + \left\lfloor \frac{2012}{8} \right\rfloor + \ldots + \left\lfloor \frac{2012}{1024} \right\rfloor = \sum_{k=1}^{10} \left\lfloor \frac{2012}{2^k} \right\rfloor$$

(2)

Note that $\left\lfloor \frac{2012}{4} \right\rfloor = \left\lfloor \frac{1}{2} \left\lfloor \frac{2012}{2} \right\rfloor \right\rfloor$, so that in calculating of $\left\lfloor \frac{2012}{2^{k+1}} \right\rfloor$ we can use our knowledge of $\left\lfloor \frac{2012}{2^k} \right\rfloor$, i.e. that

$$\left\lfloor \frac{2012}{2^{k+1}} \right\rfloor = \left\lfloor \frac{1}{2} \left\lfloor \frac{2012}{2^k} \right\rfloor \right\rfloor.$$

Using this formula repeatedly, we obtain

$$\left\lfloor \frac{2012}{2} \right\rfloor = 1006, \left\lfloor \frac{2012}{4} \right\rfloor = \left\lfloor \frac{1006}{2} \right\rfloor = 503, \left\lfloor \frac{2012}{8} \right\rfloor = \left\lfloor \frac{503}{2} \right\rfloor = 251,$$

$$\left\lfloor \frac{2012}{16} \right\rfloor = \left\lfloor \frac{251}{2} \right\rfloor = 125, \left\lfloor \frac{2012}{32} \right\rfloor = \left\lfloor \frac{125}{2} \right\rfloor = 62, \left\lfloor \frac{2012}{64} \right\rfloor = \left\lfloor \frac{62}{2} \right\rfloor = 31,$$
\[
\left[ \frac{2012}{128} \right] = \left[ \frac{31}{2} \right] = 15, \quad \left[ \frac{2012}{256} \right] = \left[ \frac{15}{2} \right] = 7, \quad \left[ \frac{2012}{512} \right] = \left[ \frac{7}{2} \right] = 3, \quad \left[ \frac{2012}{1024} \right] = \left[ \frac{3}{2} \right] = 1.
\]

Hence

\[ n = 1006 + 503 + 251 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 2004. \]

The answer is 2004.

3. What is the sum of all real numbers \( x \) that satisfy

\[(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3?\]

Solution

Let \( a = 2^x - 4 \) and \( b = 4^x - 2 \), then our equation can be written as \( a^3 + b^3 = (a + b)^3 \). Using the formula \( a^3 + b^3 = (a + b)(a^2 - ab + b^2) \) we get that

\[(a + b)((a + b)^2 - (a^2 - ab + b^2)) = 3ab(a + b) = 0.\]

So, either \( a = 0 \) or \( b = 0 \) or \( a + b = 0 \).

(a) \( a = 0 \iff 2^x = 4 \iff x = 2 \);
(b) \( b = 0 \iff 4^x = 2 \iff x = \frac{1}{2} \);
(c) \( a + b = 0 \iff 2^x + 4^x - 6 = 0 \iff (2^x)^2 + 2^x - 6 = 0 \). Let \( y = 2^x \). Then \( y^2 + y - 6 = (y + 3)(y - 2) = 0 \). Since \( y = 2^x > 0 \) we get that \( y = 2^x = 2 \), therefore \( x = 1 \).

Summing up the solutions obtained in all 3 cases above we get \( 2 + \frac{1}{2} + 1 = \frac{7}{2} \).

The answer is \( \frac{7}{2} \).

4. On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd numbers of dots?

Solution

The total number of dots on the die is \( 1 + 2 + 3 + \ldots + 6 = \frac{6 
\times 7}{2} = 21 \). The number of dots on the faces with an odd number of dots is \( 1 + 3 + 5 = 9 \) and the number of dots on the faces with an even number of dots is equal to \( 21 - 9 = 12 \). There are two cases:
(a) The dot is removed from a face with an odd number of dots. By above, the probability of this is $\frac{9}{21} = \frac{3}{7}$. In this case we get 2 (out of 6) faces with odd number of dots, so under the condition that the dot is removed from the face with an odd number of dots the probability that the top face is odd is $\frac{2}{6} = \frac{1}{3}$. Therefore the probability that the dot is removed from the face with an odd number of dots and the top face will be odd is $\frac{3}{7} \cdot \frac{1}{3} = \frac{1}{7}$.

(b) The dot is removed from a face with an even number of dots. By above, the probability of this is $\frac{12}{21} = \frac{4}{7}$. In this case we get 4 (out of 6) faces with odd number of dots, so under the condition that the dot is removed from the face with an even number of dots the probability that the top face is odd is $\frac{4}{6} = \frac{2}{3}$. Therefore the probability that the dot is removed from the face with an even number of dots and the top face will be odd is $\frac{4}{7} \cdot \frac{2}{3} = \frac{8}{21}$.

Therefore the probability that the top face will be odd is equal to the sum of the probabilities of cases (a) and (b), i.e. $\frac{1}{7} + \frac{8}{21} = \frac{11}{21}$.

The answer is $\frac{11}{21}$.

5. It is known that the number $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}$ is a rational number. Find this number (in term of fraction in lowest term).

Solution

Multiply and divide our expression by $\sin \frac{\pi}{7}$, then use consecutively the formula for the sin of a double angle:

$$
\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = \frac{\sin \frac{\pi}{7} \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}}{\sin \frac{\pi}{7}} = \frac{\sin \frac{2\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}}{2 \sin \frac{\pi}{7}} = \frac{\sin \frac{4\pi}{7} \cos \frac{4\pi}{7}}{4 \sin \frac{\pi}{7}} = \frac{\sin \frac{8\pi}{7}}{8 \sin \frac{\pi}{7}}
$$

Note that $\sin \frac{8\pi}{7} = \sin(\pi + \frac{\pi}{7}) = -\sin \frac{\pi}{7}$. Therefore,
\[
\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = \frac{-\sin \frac{\pi}{7}}{8 \sin \frac{\pi}{7}} = -\frac{1}{8}.
\]

The answer is \(-\frac{1}{8}\).

6. The number \(25^{64} \cdot 64^{25}\) is the square of a positive integer \(N\). What is the sum of the decimal digits of \(N\)?

Solution

\(N = 5^{64} \cdot 8^{25} = 5^{64} \cdot 2^{75} = 5^{64} \cdot 2^{64} \cdot 2^{11} = 2^{11} \cdot 10^{64} = 2048 \cdot 10^{64}\). therefore the sum of the decimal digits of \(N\) is equal to the sum of the decimal digits of 2048, i.e. is equal to 14

The answer is 14

7. How many nonequivalent ways \(n\) individuals can be seated at the round table, if two ways are considered equivalent when each person has the same neighbors.

Solution If we assume that the \(n\) person are seated in the line, the number of ways is equal to the number of permutations of \(n\) elements, i.e. to \(n!\). However, if they are seated in the round table we can rotate a given permutation or flip it to get an equivalent configuration, so that the set of all permutations is subdivided by the subsets of equivalent permutations (called equivalence classes). The number of permutations in each equivalence class is equal to \(2n\) because starting from some permutation we can rotate it \(n-1\) times (obtaining in this way \(n-1\) permutations equivalent to the original one) and we can flip the original permutation and rotate the obtained permutation \(n-1\) times (getting another \(n\) equivalent permutation to the original one). Therefore the required number of ways is equal to the number of the equivalence classes and it is equal to \(\frac{n!}{2n} = \frac{1}{2}(n-1)!\).

The answer is \(\frac{1}{2}(n-1)!\).

8. The polynomial \(P(x) = x^3 + ax^2 + bx + c\) has the property that the average of its zeros, the product of its zeros, and the sum of its coefficients are all equal. The \(y\)-intercept of the graph \(y = P(x)\) is 5. What is \(b\)?

Solution Since the \(y\)-intercept of the graph \(y = P(x)\) is equal to \(P(0) = c\), we get that \(c = 5\). By the Vieta Theorem the product of the zeros of \(P(x)\) is equal to \(-c\), i.e. to \(-5\). By the same theorem the sum of the zeros of \(P(x)\), is equal to \(-a\), therefore their average is equal to \(-a/3\). Therefore \(-a/3 = -5\), i.e \(a = 15\). Further,
the sum of coefficient of $P(x)$ is equal to $1 + a + b + c = 1 + 15 + b + 5 = 21 + b$. On the other hand, by the conditions this sum is equal to $-5$. So $21 + b = -5$, therefore $b = -5 - 21 = -26$.

**The answer is** $-26$.

9. A point $P$ is selected at random from the interior of the pentagon with the vertices $A(0, 2)$, $B(4, 0)$, $C(2\pi+1, 0)$, $D = (2\pi+1, 4)$, and $E = (0, 4)$. What is the probability that $\angle APB$ is obtuse.

**Solution** Draw the semicircle inside the pentagon with the diameter $AB$ (see the picture).

Then for the interior point $P$ the angle $\angle APB$ is obtuse if and only if $P$ is inside of this semicircle. Therefore the desired probability is the ratio of the area of the corresponding semidisk and the area of the pentagon. The radius of the semicircle is $\sqrt{5}$, therefore the area of the semidisk is $\frac{5}{2}\pi$. The area of the pentagon is equal to the sum of the area of the rectangle $ADEF$ and the trapezoid $AFCB$, i.e. $|AE| \cdot |ED| + \frac{|ED| + |BC|}{2} |CF| = 2(2\pi + 1) + \frac{2\pi + 1 + 2\pi - 3}{2} \cdot 2 = 8\pi$. Therefore the desired probability $= (\frac{5}{2}\pi) : (8\pi) = \frac{5}{16}$.

**The answer is** $\frac{5}{16}$.

10. It is known that the number

$$\sqrt{\frac{11 \ldots 1}{2012} - \frac{22 \ldots 2}{1006}}$$

is integer. Find this integer in terms of its decimal representation. Here by $11 \ldots 1$ we mean the number that has the decimal representation consisting of the digits 1 repeated 2012 times and by $22 \ldots 2$ the number that have the decimal representation consisting of the digits 2 repeated 1006 times.
Solution Using the formula for the sum of geometric progression we get

\[
\underbrace{11 \ldots 1}_{2012 \text{ times}} = 1 + 10 + \ldots 10^{2011} = \frac{10^{2012} - 1}{9}
\]

\[
\underbrace{22 \ldots 2}_{1006 \text{ times}} = 2 \cdot \frac{10^{1006} - 1}{9}
\]

Therefore

\[
\sqrt{\underbrace{11 \ldots 1}_{2012 \text{ times}} - \underbrace{22 \ldots 2}_{1006 \text{ times}}} = \sqrt{\frac{10^{2012} - 2 \cdot 10^{1006} + 1}{3}} = \frac{10^{1006} - 1}{3} = \frac{10^{1006} - 1}{9} = \underbrace{33 \ldots 3}_{1006 \text{ times}}
\]

The answer is \(\underbrace{33 \ldots 3}_{1006 \text{ times}}\).

11. An ant has one sock and one shoe for each of its six legs. In how many different orders can the ant put on its socks and shoes, assuming that, on each leg, the sock must be put on before a shoe?

Solution

Number the ant’s legs from 1 to 6 and let \(a_k\) and \(b_k\) denote the sock and the shoe that will go on \(k\)th leg. We have 12 object and without the restriction that on each leg, the sock must be put on before a shoe we have \(12!\) possible orders. In half of them \(a_1\) precedes \(b_1\), in half of the latter also \(a_2\) precedes \(b_2\), and so on. So the number of orders satisfying our conditions is equal to \(\frac{12!}{2^6}\).

The answer is \(\frac{12!}{2^6}\).

12. Circles \(A, B, C\) are externally tangent to each other and internally tangent to circle \(D\). Circles \(B\) and \(C\) are congruent. Circle \(A\) has radius 1 and passes through the center of circle \(D\). What is the radius of circle \(B\)?
Solution Let $F$ be the center of the big circle $D$, $E$ is the center of circle $A$, $H$ is the center of circle $B$, and $G$ be the points of tangency of the congruent circles $B, C$ as in the figure. Then if $r$ is the radius of circle $B$ we can find the equation for $r$ from the triangle $EFH$, using the Cosine Theorem. Indeed, $|EF| = 1$, $|EH| = 1 + r$, and $|FH| = 2 - r$ (here we use that the radius of the big circle $D$ is equal to 2). Further,

$$\cos(\angle EFH) = -\cos(\angle GFH) = -\frac{\sqrt{(2 - r)^2 - r^2}}{2 - r} = -\frac{\sqrt{4 - 4r}}{2 - r} = -\frac{2\sqrt{1 - r}}{2 - r}$$

from the right triangle $GFH$. Therefore by the Cosine theorem:

$$(1 + r)^2 = (2 - r)^2 + 1 + 2(2 - r)\frac{2\sqrt{1 - r}}{2 - r} = (2 - r)^2 + 1 + 4\sqrt{1 - r}$$

Simplifying we get the equation

$$3r - 2 = 2\sqrt{1 - r}$$

Taking the square of both sides of it we get $9r^2 - 12r + 4 = 4 - 4r \Leftrightarrow 9r^2 - 8r = 0 \Leftrightarrow r = \frac{8}{9}$ or 0. Since the radius is positive we get

The answer is $\frac{8}{9}$.

13. Find the sum of all coefficients of the polynomial obtained after expanding and collecting the like terms of the product

$$(1 - 2x + 3x^2)^{2012}(1 + 3x - 2x^2)^{2013}$$

Solution The sum of the coefficients of a polynomial $P(x)$ is equal to $P(1)$. In our case plugging $x = 1$ we get $(1 - 2 + 3)^{2012}(1 + 3 - 2)^{2013} = 2^{2012}2^{2013} = 2^{4025}$.

The answer is $2^{4025}$.
14. Find all real solutions of the equation

\[ 2 \sin^3 \left( \frac{3x}{2} \right) \cos^4 4x = \frac{\pi^2}{x^2} + \frac{x^2}{\pi^2} \]

**Solution** First of all recall that for any real \( y \)

\[ y^2 + \frac{1}{y^2} \geq 2 \]

and the equality holds if and only if \( y^2 = 1 \) This is because \( y^2 + \frac{1}{y^2} - 2 = (\frac{1}{y} - y)^2 \geq 0 \).

Using this with \( y = \frac{x}{\pi} \) we get that the right hand side of our equation is greater or equal than 2 and the equality holds if and only if \( x^2 = \pi^2 \), i.e. \( x = \pi \) or \( x = -\pi \).

On the other hand, since the absolute values of sin and cos are bounded by 1 the left hand side of our equation is not greater than 2. Therefore, if \( x \) is a solution then \( x \) is either \( \pi \) or \( -\pi \). Let us check whether \( \pi \) and \( -\pi \) are solutions by plugging them into the left hand side of the equation (the right hand side is equal to 2 in both cases). Plugging \( x = \pi \) we get \( 2 \sin^3 \left( \frac{3\pi}{2} \right) \cos^4 4\pi = -2 \), so \( x = \pi \) is not a solution. Plugging \( x = -\pi \) we get \( 2 \sin^3 \left( -\frac{3\pi}{2} \right) \cos^4 (-4\pi) = 2 \), so \( x = -\pi \) is the unique solution.

The answer is \(-\pi\)

15. Find

\[ \lim_{x \to 0} (\cos x)^{1/x^2} \]

**Solution**

**Way 1**

\[ \lim_{x \to 0} (\cos x)^{1/x^2} = \lim_{x \to 0} (1 + (\cos x - 1))^{1/x^2} = \lim_{x \to 0} \left( 1 + (\cos x - 1) \right)^{\frac{\cos x - 1}{x^2}} \]

Note that using the substitution \( y = \cos x - 1 \) we get
\[
\lim_{x \to 0} (1 + (\cos x - 1))^{1/(\cos x - 1)} = \lim_{y \to 0} (1 + y)^{1/y} = e \quad (4)
\]

Also
\[
\lim_{x \to 0} \frac{\cos x - 1}{x^2} = -2 \lim_{x \to 0} \frac{\sin^2\left(\frac{x}{2}\right)}{x^2} = -\frac{2}{4} \lim_{x \to 0} \frac{\sin^2\left(\frac{x}{2}\right)}{(x/2)^2} = -1 \left( \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{x/2} \right)^2 = -\frac{1}{2} \quad (5)
\]

Then using (4) and (5) in (3) we get \(\lim_{x \to 0} (\cos x)^{1/x^2} = e^{-1/2}\)

**Way 2**
\[
\lim_{x \to 0} (\cos x)^{1/x^2} = \lim_{x \to 0} e^{\frac{\ln(\cos x)}{x^2}} \quad (6)
\]

Then using L'Hopital rule we have
\[
\lim_{x \to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x \to 0} \frac{-\sin x}{2x} = -\frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x \cos x} = -\frac{1}{2}
\]

Using this in (6) we get \(\lim_{x \to 0} (\cos x)^{1/x^2} = e^{-1/2}\)

**The answer is** \(e^{-1/2}\).

16. Find the sum
\[
1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n!.
\]

**Solution**

Note that
\[
(n + 1)! - n! = (n + 1)n! - n! = n \cdot n!
\]

Therefore we get the telescopic sum:
\[
1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (2! - 1!) + (3! - 2!) + \ldots + ((n + 1)! - n!) = (n + 1)! - 1.
\]

**The answer is** \((n + 1)! - 1\).
17. Calculate the integral

$$\int_0^{\pi/2} \cos^2(\cos x) + \sin^2(\sin x) \, dx.$$ 

**Solution**

Using the substitution $x = \frac{\pi}{2} - y$, we get that

$$\int_0^{\pi/2} \sin^2(\sin x) \, dx = \int_0^{\pi/2} \sin^2(\sin(\frac{\pi}{2} - y)) \, dy = \int_0^{\pi/2} \sin^2(\cos(y)) \, dy = \int_0^{\pi/2} \sin^2(\cos x) \, dx.$$ 

Therefore,

$$\int_0^{\pi/2} \cos^2(\cos x) + \sin^2(\sin x) \, dx = \int_0^{\pi/2} \cos^2(\cos x) \, dx + \int_0^{\pi/2} \sin^2(\sin x) \, dx =$$

$$\int_0^{\pi/2} \cos^2(\cos x) \, dx + \int_0^{\pi/2} \sin^2(\cos x) \, dx = \int_0^{\pi/2} \cos^2(\cos x) \, dx + \sin^2(\cos x) \, dx = \int_0^{\pi/2} \, dx = \frac{\pi}{2}.$$ 

**The answer is** $\frac{\pi}{2}$.

18. A plane contains points $A$ and $B$ with the distance between them equal to 1. Let $S$ be the union of all disks of radius 1 in the plane that cover the segment $\overline{AB}$. What is the area of $S$?

**Solution**

A point $O$ is the center of a disk that covers the segment $\overline{AB}$ if and only the distance from $O$ to both $A$ and $B$ does not exceed 1. In other words the set $R$ of all centers of the disks of radius 1 that cover the segment $\overline{AB}$ is equal to the intersection of two discs of radius 1 centered at $A$ and $B$ (see the first figure).
The desired set $S$ is the set of all points such that the distance from them to the set $R$ does not exceed 1. To describe the set $S$ let $C$ and $D$ be the points of intersections of the unit circles with centers at $A$ and $B$. Consider two sectors of the circles of radius 2 generated by the angles $\angle CAD$ and $\angle CBD$. Consider also two sectors of the circles of radius 1 with centers at $C$ and $D$ such that the first sector is bounded by lines $AC$ and $BC$ and it is out of the set $R$ and the second sector is bounded by lines $AD$ and $BD$ and out of the set $R$ (see the second figure). Then the case by case analysis shows that the union of these four sectors belongs to $S$ and any point out of this union does not belong to $S$, so that $S$ coincides with this union.

It remains to calculate the area of the union of these four sectors. The two sectors of radius 4 have the angle $\frac{2\pi}{3}$, therefore the area of each of them is equal to $\frac{\pi}{3^2} = \frac{4\pi}{3}$. Their intersection is the rhombus consisting of two equilateral triangles with the side of length 1 and the area of this rhombus is equal to $2\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$. Therefore the area of the union of this two sectors is equal to $8\frac{\pi}{3} - \frac{\sqrt{3}}{2}$. The remaining two sectors of radius 1 have the angle $\frac{\pi}{3}$, therefore the area of each of them is equal to $\frac{\pi}{6}$. They do not have common points with each other and with two other sectors, except points in the boundaries. Therefore the total area of $S$ is equal to $\frac{8\pi}{3} - \frac{\sqrt{3}}{2} + 2\frac{\pi}{6} = 3\pi - \frac{\sqrt{3}}{2}$.

**The answer is** $3\pi - \frac{\sqrt{3}}{2}$.

19. The increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that $a_{n+2} = a_n + a_{n+1}$, for all $n \geq 1$. Suppose that $a_7 = 120$. What is $a_8$?
Solution Using repeatedly the recursive formula $a_{n+2} = a_n + a_{n+1}$ it is easy to show that $a_7 = 5a_1 + 8a_2$. Hence

$$5a_1 + 8a_2 = 120. \tag{7}$$

Then $5a_1 = 120 - 8a_2 = 8(15 - a_2)$. Since 5 and 8 are relatively prime we get from here that $a_1$ is divisible by 8, i.e. $a_1 = 8i$ for some natural $i$. In the same way $8a_2 = 120 - 5a_1$, which implies that $a_2$ is divisible by 5, i.e. $a_2 = 5j$ for some natural $j$. Therefore (7) can be written as $40i + 40j = 120$ which implies that $i + j = 3$. So we have two possibilities:

(a) $i = 1, j = 2$. Then $a_1 = 8$ and $a_2 = 10$. Note that from the recursive relation we can show again that $a_8 = 8a_1 + 13a_2$. Therefore $a_8 = 8 \cdot 8 + 13 \cdot 10 = 64 + 130 = 194$.

(b) $i = 2, j = 1$. Then $a_1 = 16$ and $a_2 = 5$, which contradicts the assumption that our sequence is increasing.

The answer is 194.

20. Suppose that $n$ is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors. How many positive divisors does $6n$ have?

Solution

Given a prime number $p$ denote by $\alpha_p$ the maximal integer such that $p^{\alpha_p}$ divides the number $n$. Then

$$n = 2^{\alpha_2}3^{\alpha_3}5^{\alpha_5}\ldots \text{ and}$$

$$\begin{align*}
\# \text{ of divisors of } n &= (\alpha_2 + 1)(\alpha_3 + 1)(\alpha_5 + 1)\ldots \tag{8} \\
\# \text{ of divisors of } 2n &= (\alpha_2 + 2)(\alpha_3 + 1)(\alpha_5 + 1)\ldots = 28 \tag{9} \\
\# \text{ of divisors of } 3n &= (\alpha_2 + 1)(\alpha_3 + 2)(\alpha_5 + 1)\ldots = 30 \tag{10}
\end{align*}$$

Subtracting (9) from (10), we get

$$2 = ((\alpha_2 + 1)(\alpha_3 + 2) - (\alpha_2 + 2)(\alpha_3 + 1))(\alpha_5 + 1)\ldots = (\alpha_2 - \alpha_3)(\alpha_5 + 1)\ldots$$

Therefore, we have two possibilities:

(a) $\alpha_2 - \alpha_3 = 1$ and $(\alpha_5 + 1)\ldots = 2$. Then $\alpha_2 = \alpha_3 + 1$ and from (10) we have that $30 = (\alpha_3 + 2)^2 \cdot 2 \rightarrow (\alpha_3 + 2)^2 = 15 \Rightarrow$ contradiction.
(b) \( \alpha_2 - \alpha_3 = 2 \) and \( (\alpha_3+1) \ldots = 1 \). Then \( \alpha_2 = \alpha_3 + 2 \) and from (10) \( 30 = (\alpha_2+1)\alpha_2 \) which implies that \( \alpha_2 = 5 \) and \( \alpha_3 = 3 \) and all \( \alpha_p = 0 \) for \( p \geq 5 \). Note that this implies (9) as well. Further \( n = 2^5 \cdot 3^3 \). Hence \( 6n = 2^6 \cdot 3^4 \) and the number of divisors of \( 6n \) is equal to \( 7 \cdot 5 = 35 \).

The answer is 35.