1. If \(a + b + c = 3\), \(ab + bc + ac = 4\), and \(abc = -10\), find the value of \(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\).

**Solution:** We have that
\[
16 = (ab+bc+ac)^2 = (ab)^2 + (bc)^2 + (ac)^2 + 2abc(a+b+c).
\]
Hence, \(\Delta := (ab)^2 + (bc)^2 + (ac)^2 = 16 - 2 \cdot (-10) \cdot 3 = 76\). The we derive
\[
\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{\Delta}{(abc)^2} = \frac{76}{100} = \frac{19}{25}.
\]
Answer: \(\frac{19}{25}\)

2. A standard die is rolled three times. Find the probability that the product of the outcomes is divisible by 8.

**Solution:** Let \(x, y, z\) be the outcomes of rolls one, two and three respectively. The total number of outcomes for three rolls is \(T = 6^3 = 216\). Let \(G\) be the number of outcomes in which \(xyz\) is divisible by 8. We need to find the probability \(P = \frac{G}{T}\). There are two ways to get divisibility by 8: (i) all \(x, y, z\) are even numbers; (ii) one of \(x, y, z\) is an odd number. The number of ways to get (i) is \(3 \cdot 3 \cdot 3 = 27\). In the second case, we can assume the first number is odd, find all outcomes divisible by 8, and then multiply that by 3. Then \((x, y, z)\) must be of the form (odd, even, even) with \(y\) or \(z\) equal to 4. This can be done in \(3 \cdot 5 = 15\) ways. Therefore \(G = 3 \cdot 15 + 27 = 72\) and the probability is \(P = \frac{1}{3}\).

Answer: \(\frac{1}{3}\)

3. In how many ways the letters T, A, M, and U can be arranged in a sequence so that A is not in position 3, U is not in position 1, T is not in position 2, and M is not in position 4?

**Solution:** If \(T\) is in position 1, then \(U\) can be in any of the other three positions and there is a unique way to place the other two letters. Similarly, if \(T\) is in position 3, then \(A\) can be in any of the other three positions and the rest of the letters have predetermined positions. Finally, if \(T\) is in position 4, then \(M\) can be in any of the other three positions and the rest of the order is fixed. This makes the total number of arrangement \(3 + 3 + 3 = 9\).
4. If $\theta$ is an acute angle and $\sin \theta = a$ then find $\sin \frac{\theta}{2}$ in terms of $a$.

Solution: Let $x = \sin \frac{\theta}{2}$. We use that $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ to derive $a^2 = 4x^2(1 - x^2)$. Because $\theta$ is an acute angle, we derive $0 < x < \sin \frac{\pi}{4}$ which implies $0 < x^2 < \frac{1}{2}$. The only solution to the equation $4x^4 - 4x^2 - a^2 = 0$ with these properties is $x = \sqrt{\frac{1 - \sqrt{1 - a^2}}{2}}$.

Answer: $x = \sqrt{\frac{1 - \sqrt{1 - a^2}}{2}}$

5. Let $f$ be a function such that $f(x) + 2f(2010 - x) = 10x + 23$ for every real number $x$. What is the value of $f(1000)$?

Solution: We take $x = 1000$ and $x = 1010$ in the equation for $f$ and get: $f(1000) + 2f(1010) = 10023$ and $f(1010) + 2f(1000) = 10123$. Solving for $f(1000)$, we obtain $f(1000) = \frac{10223}{3} = 3407\frac{2}{3}$

Answer: $3407\frac{2}{3}$

6. Find all real numbers $x$ which satisfy $\log_x 81 - \log_3 x^4 = 6$.

Solution: Using properties of logarithm we get $4 \log_x 3 - 4 \log_3 x = 6$. Let $y = \log_3 x$, then $y$ satisfies $2y^2 + 3y - 2 = 0$. The roots of the quadratic equation are $y = 1/2$ and $y = -2$. This gives $x = 1/9$ and $x = \sqrt{3}$ as solutions.

7. Find the area of the region in the $xy$-plane consisting of all points $(x, y)$ whose coordinates satisfy the inequalities $||y| - |x|| \leq 2$ and $|y - 1| \leq 4$.

Solution: We first observe that if $(x, y)$ is in this region then $(-x, y)$ is also there. Therefore, it is enough to compute the area of the part where $x > 0$ and double it.

The restrictions on the “half” region $D$ are: $-2 \leq |y| - x \leq 2$ and $-3 \leq y \leq 5$. The lines forming the boundaries of $D$ are $y = x + 2$, $y = -x - 2$, $y = x - 2$, $y = -x + 2$, $y = 5$, $y = -3$ and $x = 0$. It easy to verify that $D$ is composed of three parts: a right triangle with vertices $(0, -2)$, $(2, 0)$, and $(0, 2)$; a trapezoid with vertices $(0, 2)$, $(3, 5)$, $(7, 5)$, and $(2, 0)$; a trapezoid with vertices $(0, -2)$, $(1, -3)$, $(5, -3)$, and $(2, 0)$. Then the area of $D$, $|D|$ is the sum of the three areas

$$|D| = \frac{1}{2} \cdot 2\sqrt{2} \cdot 2\sqrt{2} + 2\sqrt{2} \cdot \frac{3\sqrt{2} + 5\sqrt{2}}{2} + 2\sqrt{2} \cdot \frac{3\sqrt{2} + \sqrt{2}}{2} = 28.$$
Then the total area of the region is $2|D| = 56$.

Answer: 56

8. Express $\tan \left( \arctan \frac{1}{2} + \arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{5} \right)$ as a rational number in lowest terms.

**Solution:** Let $\tan \alpha = \frac{1}{2}$, $\tan \beta = \frac{1}{3}$, $\tan \gamma = \frac{1}{4}$, and $\tan \delta = \frac{1}{5}$. Using the formula for $\tan(x + y)$, we get

$$\tan(\alpha + (\beta + \gamma + \delta)) = \frac{\frac{1}{2} + \tan(\beta + \gamma + \delta)}{1 - \frac{1}{2} \tan(\beta + \gamma + \delta)}.$$

Similarly, we have

$$\tan(\beta + \gamma + \delta) = \frac{\frac{1}{3} + \tan(\gamma + \delta)}{1 - \frac{1}{3} \tan(\gamma + \delta)}, \quad \text{and} \quad \tan(\gamma + \delta) = \frac{\frac{1}{4} + \frac{1}{5}}{1 - \frac{1}{20}} = \frac{9}{19}.$$

Using backwards substitution we obtain $\tan(\beta + \gamma + \delta) = \frac{23}{24}$ and we conclude

$$\tan(\arctan \frac{1}{2} + \arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{5}) = \frac{14}{5}.$$

9. Find all ordered pairs of real numbers $(a, b)$ such that $\sqrt{a + b} = \sqrt{a} + 1$ and $0.25b^2 - 2a = 0.5$.

**Solution:** Using the first equation, we have $a + b = (\sqrt{a} + 1)^2$ and this implies $a = \frac{(b-1)^2}{4}$. We use this in the second equation and derive $b^2 - 4b + 4 = 0$. Therefore, the only solution is the pair $(a, b) = (\frac{1}{4}, 2)$.

Answer: $\left( \frac{1}{4}, 2 \right)$

10. On a white piece of paper a large circle is drawn. The large circle contains an inscribed square. Inside the inscribed square there is a circle. Two perpendicular lines divide this circle into four parts. Four smaller circles are inscribed into each one of those parts. The small circles each have a radius of length 1. Using this information, find the length of the boundary of the shaded portion shown in the figure below.
Solution: The boundary of the shaded part consists of an outside circle with a radius \( R \), a square inscribed in that circle with a side \( a \), an inner circle with a radius \( r \), and four small circles each with a radius equal to 1. It is easy to see that \( a = \sqrt{2}R \), \( a = 2r \), and \( r = 1 + \sqrt{2} \). Therefore the total length \( L \) of the boundary of the shaded part is

\[
L = 2\pi(2 + \sqrt{2}) + 4((2\sqrt{2} + 4) + 2\pi(\sqrt{2} + 1) + 4 \cdot 2\pi.
\]

This simplifies to

\[
L = 8 + 8\sqrt{2} + 14\pi + 4\sqrt{2}\pi.
\]

Answer: \( 8 + 8\sqrt{2} + 14\pi + 4\sqrt{2}\pi \).

11. A team contains five boys and five girls. For the team banquet, they select seats at random around a circular table that seats ten people. What is the probability that some two girls will seat next to one another?

Solution: We will compute the probability of the complement. That is, we will find the probability \( p \) of the event that no two girls are adjacent. Then the answer for the probability that some two girls will seat next to one another is going to be \( 1 - p \). The total number of placements of ten people in a circle is \( 9! \). If we fix the position of one girl, then we must have boys of either side of her. Moreover, on the other seven places left we have no other option but to have an alternating pattern: girl, boy, . . . . Then, we have that there are only five seats for boys and four for girls. Hence \( p = \frac{4!5!}{9!} \) and this gives that \( 1 - p = \frac{125}{126} \).

12. Find \( \sum_{i=1}^{10} a_i^2 \), where \( a_n = n + \frac{1}{2n + \frac{1}{n+\ldots}}. \)

Solution: Note that \( a_n = n + \frac{1}{n+a_n} \) and this gives \( a_n^2 = n^2 + 1. \) then the sum simplifies to

\[
\sum_{i=1}^{10} a_i^2 = \sum_{i=1}^{10} i^2 + 10 = 395.
\]

Answer: 395

13. In triangle \( ABC \), see Figure 1, segments \( CE \) and \( AD \) are drawn so that \( \frac{CD}{DB} = 3 \) and \( \frac{AE}{EB} = 1.5 \). Let \( r = \frac{CP}{PE} \), where \( P \) is the intersection point of \( CE \) and \( AD \). Find \( r \).

Solution: Menelaus’s Theorem gives \( \frac{EP}{PC} \cdot \frac{CD}{DB} \cdot \frac{BA}{AE} = 1. \) Using the information given we get \( \frac{3}{r} \frac{AE+EB}{AE} = 1. \) Then the equation for \( r \) is \( \frac{3}{r}(1 + \frac{2}{3}) = 1 \) which has a solution \( r = 5. \)
14. Three relatively prime integers are the sides of a right triangle. If the smallest side of the triangle has length 28, find the sum of all possible values of the hypotenuse.

**Solution:** Euclid’s formula gives that the sides of this triangle form a Pythagorean triple: \( m^2 + n^2, 2mn, \) and \( m^2 - n^2, \) where \( m, n \) are positive integers with \( m > n. \) The only even number in this triple is \( 2mn = 28. \) The only two possibilities for \( (m, n) \) are \((7, 2)\) and \((14, 1)\). These two pairs give the values 53 and 197 for the hypotenuse. Their sum is 250.

Answer: 250

15. Solve the equation 
\[
1! + 2! + 3! + \cdots + x! = y^2
\]
for all integer pairs \((x, y)\).

**Solution:** If \( x \geq 5 \) then \( 1! + 2! + 3! + \cdots + x! \) has a last digit 3 and no perfect square can have a last digit 3. Hence, we are left with checking the options \( x = 1, 2, 3, 4. \) The only two solutions for \( (x, y) \) are \((1, 1)\) and \((3, 3)\).

Answer: \((1, 1), (3, 3)\)

16. Find the greatest integer less than \( \left( \sqrt{5} + \sqrt{7} \right)^6. \)

**Solution:** Note that the sum \( \left( \sqrt{7} + \sqrt{5} \right)^6 + \left( \sqrt{7} - \sqrt{5} \right)^6 \) is an integer and after
a calculation we get

\[
\left(\sqrt{7} + \sqrt{5}\right)^6 + \left(\sqrt{7} - \sqrt{5}\right)^6 = 13536.
\]

Moreover, \(0 < (\sqrt{7} - \sqrt{5}) < 1\) and this gives

\[
13536 - 1 < \left(\sqrt{5} + \sqrt{7}\right)^6 < 13536.
\]

Then the greatest integer less than \(\sqrt{5} + \sqrt{7}\) is 13535.

Answer: 13535

17. Find all ordered pairs \((m, n)\) where \(m\) and \(n\) are positive integers such that \(\frac{n^3+1}{mn-1}\) is an integer.

**Solution:** Suppose \(\frac{n^3+1}{mn-1} = k\) where \(k\) is a positive integer. Then \(n^3 + 1 = (mn - 1)k\) and so it is clear that \(k \equiv -1 \pmod{n}\). So, let \(k = jn - 1\) where \(j\) is a positive integer. Then we have \(n^3 + 1 = (mn - 1)(jn - 1) = mjn^2 - (m + j)n + 1\) which by canceling out the 1s and dividing by \(n\) yields \(n^2 = mjn - (m + j)\) or \(n^2 - mjn + m + j = 0\). The equation \(x^2 - mjr + m + j = 0\) is a quadratic. We are given that \(n\) is one of the roots. Let \(p\) be the other root. Notice that since \(n + p = mj\) we have that \(p\) is an integer, and so from \(np = m + j\) we have that \(p\) is positive.

It is obvious that \(j = m = n = p = 2\) is a solution. Now, if not, and \(j, m, n, p\) are all greater than 1, we have the inequalities \(np > n + p\) and \(mj > m + j\) which contradicts the equations \(np = m + j, n + p = mj\). Thus, at least one of \(j, m, n, p\) is equal to 1.

If one of \(m, j\) is 1, without loss of generality assume it is \(j\). Then we have \(np = m + 1, n + p = m\). That is, \(np - n - p = 1\) or \((n - 1)(p - 1) = 2\) which gives positive solutions \((n, p) = (3, 2), (2, 3)\). These give \(m = 5\) and since we assumed \(j = 1\), we can also have \(m = 1\) and \(j = 5\).

If one of \(n, p\) is 1, without loss of generality assume it is \(p\). Then we have \(n = m + j, n + 1 = mj\). That is, \(mj - m - j = 1\) or \((m - 1)(j - 1) = 2\) which gives positive solutions \((m, j) = (3, 2), (2, 3)\). These give \(n = 5\) and since we assumed \(p = 1\), we can also have \(n = 1\) and \(p = 5\). From these, we have all solutions:

\[(m, n) = (2, 2), (5, 3), (5, 2), (1, 3), (1, 2), (3, 5), (2, 5), (3, 1), (2, 1)\].

Answer: \((2, 2), (5, 3), (5, 2), (1, 3), (1, 2), (3, 5), (2, 5), (3, 1), (2, 1)\).

18. An “unfair” coin has a \(2/3\) probability of turning up heads. If the coin is tossed 7 times, what is the probability that the total number of heads is even? Express your answer as a rational number in lowest terms.
Solution: The coin could end up heads 0, 2, 4, or 6 times. Hence, the probability is
\[ p = \binom{7}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^7 + \binom{7}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^5 + \binom{7}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3 + \binom{7}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^1. \]

One could simplify the sum above or observe that
\[ p = \left(\frac{1}{3} + \frac{2}{3}\right)^7 - \left(\frac{1}{3} - \frac{2}{3}\right)^7. \]
Then \[p = \frac{1+3-7}{2} = \frac{3+1}{2\cdot3^7}\text{ which simplifies to }p = \frac{1093}{2187}.\]

Answer: \(\frac{1093}{2187}\).

19. Let \(a, b, c\) be real numbers such that \(a + b + c = 2\) and \(a^2 + b^2 + c^2 = 10\). Find the maximum possible value for \(a\).

Solution: Let’s consider first a triple \((a, b, c)\) with satisfies the two equations and such that \(a \geq b > c\). Then, for any number \(x > 0\) we have that the triple \((a, \frac{b+c}{2}, \frac{b+c}{2})\) satisfies the first restriction (the sum is 2) but it is easy to verify that the sum of the squares is less than 10. Then, we can find a number \(x > 0\) such that the triple \((a + 2x, \frac{b+c}{2} - x, \frac{b+c}{2} - x)\) satisfy both restrictions. Finding \(x\) amounts to solving a quadratic equation which always has one positive root. This new triple has a bigger maximum value that the old one. That implies that the maximum for \(a\) is obtained for triples of the type \((a, b, b)\). Now the two restrictions are: \(a + 2b = 2\) and \(a^2 + 2b^2 = 10\). Solving this with the restriction \(a \geq b\) gives \(a = \sqrt{5}\).

Answer: \(\frac{\sqrt{5}}{3}\).