1. Now $x^2 + (1/x^2) = (x + (1/x))^2 - 2 = 3^2 - 2 = 7$ and $x^4 + (1/x^4) = (x^2 + (1/x^2))^2 - 2 = 7^2 - 2 = 47$.

2. If the edge increases from $x$ to $1.3x$, then the surface area increases from $6x^2$ to $6(1.69x^2) = 10.14x^2$. This is an increase of
\[
\frac{10.14x^2 - 6x^2}{6x^2} \times 100 = \frac{4.14}{6} \times 100 = 69\%.
\]

3. Let $x$ be his original sum of money in dollars. Then $x - \frac{1}{3}x - \frac{1}{5}(x - \frac{1}{3}x) = 24$ or $\frac{4}{5}x = 24$. Thus, $x = 54$.

4. Let $x$ be the height of the shrub in feet. Then $15/20 = x/4$ or $x = 3$.

5. Now
\[
\frac{x^4 - 3ax^2 + (5a - 2)x - 16}{x - 2} = x^3 + 2x^2 + (4 - 3a)x + (6 - a) + \frac{-2a - 4}{x - 2}.
\]
So $x - 2$ is a factor if $-2a - 4 = 0$ or $a = -2$.

6. Let the side length of the rhombus equal 1. Then its altitude equals $\sqrt{3}/2$, and hence this also equals its area, and equals the diameter of the inscribed circle, which hence has area $3\pi/16$. The desired ratio is $8\sqrt{3}/(3\pi)$.

7. Notice that $(n + 1)! - n! = (n + 1)n! - n! = n \cdot n!$. Thus,
\[
(2! - 1!) + (3! - 2!) + \cdots + (8! - 7!) = 8! - 1! = 40320 - 1 = 40319,
\]
since $2!$, $3!$, $\ldots$, $7!$ cancel.

8. The right triangle in the diagram shows that $r^2 = 1 + (2 - r)^2$, which gives $4r = 5$ or $r = 5/4$. 

9. Since \( 32 < 36 \), \( 2^{35} = (2^5)^7 < (6^2)^7 = 6^{14} \). Since \( 125 < 128 \), \( 5^{15} = (5^3)^5 < (2^7)^5 = 2^{35} \).

10. The function \( f(x) \) is quadratic and the coefficient of \( x^2 \) is \( 1 + 1 - 1 + 1 + 1 = 2 > 0 \), so the graph of \( f(x) \) is an upward-pointing parabola, and the minimum is attained at its vertex. The function is also symmetric about \( x = 6 \), so the vertex must be at \( x = 6 \). We easily compute \( f(6) = 1 + 1 - 4 - 4 + 9 + 9 = 2(1 + 9 - 4) = 12 \).

11. Let \( \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = r \). Then \( \frac{xyz}{abc} = r^3 \), \( x = ra \), \( y = rb \) and \( z = rc \). It follows that \( x + y = r(a + b) \), \( y + z = r(b + c) \) and \( z + x = r(c + a) \), so that \( \frac{x + y}{a + b} = \frac{y + z}{b + c} = \frac{z + x}{c + a} = r \). The given expression is equal to \( r^3(1/r^3) = 1 \).

12. \((2^x - 3^y)(2^x + 3^y) = 55\) implies either \( 2^x - 3^y = 5 \) and \( 2^x + 3^y = 11 \), or \( 2^x - 3^y = 1 \) and \( 2^x + 3^y = 55 \). The first case says \( 2^x = 8 \) and \( 3^y = 3 \) while the second case says \( 2^x = 28 \) and \( 3^y = 27 \). The first gives \( (x, y) = (3, 1) \), while the second does not give an integer value for \( x \).

13. Multiplying one equation by the other, we have \( xy + 4 + \frac{4}{xy} = 8 \). This may be rewritten as \( 0 = (xy)^2 - 4xy + 4 = (xy - 2)^2 \). Hence \( xy = 2 \).

14. For \( a \neq 1 \),
\[
1 + a + a^2 + \cdots + a^{2010} = \frac{a^{2011} - 1}{a - 1} = \frac{2a - 2}{a - 1} = 2.
\]

15. Let \( \alpha = 3^\sqrt{9 + 4\sqrt{5}} + 3^\sqrt{9 - 4\sqrt{5}} \). Since \( (9 + 4\sqrt{5})(9 - 4\sqrt{5}) = 1 \), we obtain
\[
\alpha^3 = (9 + 4\sqrt{5}) + 3\sqrt{9 + 4\sqrt{5}} + 3\sqrt{9 - 4\sqrt{5}} + 9 - 4\sqrt{5} = 18 + 3\alpha.
\]
Thus \( \alpha \) is a root of \( x^3 - 3x - 18 = 0 \), of which \( x = 3 \) is clearly a root. In fact, \( x^3 - 3x - 18 = (x-3)(x^2+3x+6) \) and so \( 3 \) is the only real root.

16. To get the largest sum, we want the given length to be one of the legs, not the hypotenuse. If one leg has length \( A \) (in our case an integer), then we want to maximize \( x + y \), where \( x \) and \( y \) are integers satisfying \( A^2 + x^2 = y^2 \). Since \( A^2 = y^2 - x^2 = (y-x)(x+y) \), to maximize \( x + y \) we set \( y-x = 2 \) (the choice \( y-x = 1 \) will result in maximizers \( x \) and \( y \) being nonintegers). Now \( A^2 = 2(x+y) \) and then the sum of the side lengths will be \( A + x + y = A + \frac{A^2}{2} = A \left(1 + \frac{A}{4}\right) = (2010)(1006) = 2,022,060 \). We obtain \( x = \frac{A^2}{2} - 1 = 1,010,024 \) and \( y = x + 2 = 1,010,026 \). 

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17. We want to know when
\[ a + \sqrt{b} + \frac{1}{a + \sqrt{b}} = a + \frac{b + 1 + a\sqrt{b}}{a + \sqrt{b}} = 2a + \frac{b + 1 - a^2}{a + \sqrt{b}} \]
is an integer. Clearly, if \( a^2 = b + 1 \), then the last expression above is an integer. Suppose the last fraction above is an integer but \( a^2 \neq b + 1 \). Setting the fraction equal to the integer \( k \) and solving for \( \sqrt{b} \), we obtain that \( \sqrt{b} = (b + 1 - a^2 - ka)/k \) (and note that \( k \neq 0 \)). Thus, \( \sqrt{b} \) is a rational number. Since \( b \) is a positive integer, this can only happen if \( b \) is the square of an integer so that \( \sqrt{b} \) is itself a positive integer. This means that \( a + \sqrt{b} \) is an integer and, in fact, at least 2. Hence, \( a + \sqrt{b} \) will be an integer but its reciprocal will not be so that their sum will not be. In other words, \( a + \sqrt{b} \) and its reciprocal sum to an integer precisely when \( a^2 = b + 1 \). The conditions \( 1 \leq a \leq 100 \) and \( 1 \leq b \leq 100 \) together with \( a^2 = b + 1 \) imply \( 2 \leq a \leq 10 \) and, for each such \( a \), there is a unique \( b \) such that the sum of \( a + \sqrt{b} \) and its reciprocal is an integer. The answer is therefore 9.

18. Observe that
\[ \frac{a}{b} = \frac{1997}{1998} + \frac{1999}{n} = \frac{1997n + 1998 \times 1999}{1998n} . \]
Since \( a \) is divisible by 1000, \( a \) is even. It follows that \( n \) must be even. We write \( n = 2m \) and simplify the above expression for \( a/b \) to obtain
\[ \frac{a}{b} = \frac{1997m + 999 \times 1999}{1998m} . \]
Since \( a \) is even, we see that \( m \) must be odd. Since 5 divides \( a \), we also see that \( m \) cannot be divisible by 5. So we consider \( m \) now having no prime divisors in common with 10. Observe that the denominator on the right-hand side above is divisible by 2 and not 4. Also, this denominator is not divisible by 5. Thus, in order for \( a \) to be divisible by 1000, it is necessary and sufficient for \( 1997m + 999 \times 1999 \) to be divisible by 2000. We solve for \( m \) by working modulo 2000. We seek \( m \) for which
\[ 0 \equiv 1997m + 999 \times 1999 \equiv -3m + 999(-1) \pmod{2000} \]
which is equivalent to
\[ m \equiv -333 \equiv 1667 \pmod{2000} . \]
(Here, we have used that 3 and 2000 have no common prime divisors so that division by 3 in a congruence modulo 2000 is permissible.) Note that \( m \equiv 1667 \pmod{2000} \) implies that \( m \) and 10 have no common prime divisors. Hence, the condition \( m \equiv 1667 \pmod{2000} \) is a necessary and sufficient condition for \( n = 2m \) to result in a fraction \( a/b \) as in the problem with \( a \) divisible by 1000. Therefore, the smallest such positive integer is \( 2 \times 1667 = 3334 \). The sum of its digits is 13.