EF Exam

(NOTE: If units are appropriate, please include them in your answer.)

1. The expression $\sqrt{9 - 3\sqrt{8}}$ can be rewritten as $\sqrt{a} - \sqrt{b}$. Find the quotient $\frac{b}{a}$.

2. The 20th term of an arithmetic sequence is $-118$; the 30th term of the sequence is $-178$. Find the first term of the sequence.

3. Solve for $x$: $x^3 - 6x^2 + 12x - 224 = 0$.

4. The distance from line $\ell$ to the origin is 5 units. If $\ell$ has negative slope and an $x$-intercept of 10, find the $y$-intercept of $\ell$.

5. Compute $\lim_{x \to \infty} \sqrt{x^3 + 4x^2 + 7x} - x$.

6. There are $A$ horizontal tangents and $B$ vertical tangents to the graph parametrized by $x = \frac{2t}{1 + t^2}, y = \frac{1 - t^2}{1 + t^2}$. Find $A + B$. 
7. Let \( f(x) = x^3 + ax^2 + bx + c \). Given \( f(2) = -3 \) and \( x = 2 \) is a critical value, but not a relative extremum of \( f \), find \( a + b + c \).

8. Let \( f \) be a one-to-one function such that \( f(1) = 3 \), \( f(3) = 1 \), \( f'(1) = -4 \), and \( f'(3) = 2 \). If \( g = f^{-1} \), find the slope of the line tangent to \( \frac{1}{g} \) at \( x = 1 \).

9. Compute \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{n}{i^2 + n^2} \).

10. Find the area enclosed by the graph of \( 5x^2y + 6xy^2 = 10xy \).

11. By simplifying all trigonometric expressions, compute \( 3(\cos 45^\circ + i \sin 45^\circ) \cdot 5(\cos 15^\circ + i \sin 15^\circ) \), where \( i = \sqrt{-1} \).

12. Some US companies reward their overseas employees by "grossing up" their paychecks by a certain percentage to cover the income tax taken out. Note, however, that when the company increases their paycheck, the amount of taxes they have to cover increases, so the "gross up" needs to be grossed up as well, and so on. If an employee is taxed at a flat rate of 25% of their paycheck, by what percentage does the company need to gross up their paycheck to completely cover all income tax?

13. Compute \( \int_{0}^{\pi/3} \frac{dx}{5 + 4 \cos(2x)} \).
14. The **Bessel function of order 1** is defined by \( J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)2^{2n+1}} \). Let \( L(x) \) be the linear function whose graph is tangent to \( J_1 \) at \( x = 0 \). Find \( L(6) \).

15. You are given the problem of partitioning a square into \( N \) smaller squares (not necessarily congruent), where \( N \) is a positive whole number. Find the product of all values of \( N \) for which the problem has no solution.

16. You are given weighted coins \( C_1, C_2, \ldots, C_{2009} \) such that the probability of heads on coin \( C_k \) is \( \frac{1}{2k + 1} \). If all 2009 coins are tossed, find the probability that the number of heads is odd.

17. Given a line \( \ell \) and points \( P, Q \) such that \( P \) is 6 cm above \( \ell \), \( Q \) is 10 cm below \( \ell \), point \( R \) is on \( \ell \), and \( PQ = 24 \) cm. Find the maximum value of \( |PR - QR| \).

18. Given a sequence defined recursively by \( a_{n+1} = 7 - a_n \), find the sum of all possible values of \( a_1 \) such that \( \{a_n\} \) converges.

19. Find the area of the graph enclosed by the polar curve \( r = 10 \cos \theta \).

20. Given the ellipse \( x^2 - 4\sqrt{3}xy + 13y^2 - 8x + 6y = 16 \), the coordinate axes can be rotated counter-clockwise by an angle \( \theta \) so that they are parallel to the major and minor axes of the ellipse. Find the smallest positive value of \( \theta \), in radians.

21. A dart is thrown at (and hits) a square dartboard. Assuming each spot on the dartboard has an equal chance of being hit, find the probability that the dart lands at a point closer to the center of the board than any of the edges. Express your answer in the form \( \frac{a + b\sqrt{c}}{d} \) where \( a, b, c, \) and \( d \) are integers.