1. Which terms must be removed from the sum if the remaining sum is to equal 1?

Solution: It is easy to get
\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} = \frac{60}{120} + \frac{30}{120} + \frac{20}{120} + \frac{15}{120} + \frac{12}{120} + \frac{10}{120} = 1 + \frac{27}{120}.
\]
Therefore, we need to remove \( \frac{27}{120} = \frac{1}{8} + \frac{1}{10} \). The only way to make that possible is to take away \( \frac{1}{8} \) and \( \frac{1}{10} \).
Answer: \( \frac{1}{8}, \frac{1}{10} \)

2. An arbitrary circle can intersect the graph of \( y = \cos x \) in

(A) at most 1 points; (B) at most 3 points; (C) at most 5 points; (D) at most 7 points; (E) at most 9 points; (F) more than 9 points.

Solution: If we take a circle with a center on the y-axis and large enough radius we can get as many intersections as we want. For example, consider the circle \( x^2 + (y - 1000)^2 = 1000^2 \). If we take \( y = 1 \) we have that \( x^2 = 1999 \) which give \( |x| > 40 \). Therefore the circle has an arc which is inside the rectangle bounded by the lines \( y = -1, y = 1, x = -40, \) and \( x = 40 \) and \( |y| < 1 \) on that arc. This guarantees 24 intersections of this circle with the the graph of \( y = \cos x \) because \( \cos x = 1 \) for \( x = -12\pi, -10\pi, \ldots, 10\pi, 12\pi \) and \( \cos x = -1 \) for \( x = -11\pi, -9\pi, \ldots, 9\pi, 11\pi \).
Answer: \( F \)

3. Find the product and express your answer in simplest terms.

\[
P = \left( 1 - \frac{1}{2^2} \right) \left( 1 - \frac{1}{3^2} \right) \left( 1 - \frac{1}{4^2} \right) \cdots \left( 1 - \frac{1}{2011^2} \right)
\]

Solution: We have
\[
P = \frac{1 \cdot 3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{3 \cdot 5}{4^2} \cdots \frac{2009 \cdot 2011}{2010^2} \cdot \frac{2010 \cdot 2012}{2011^2} = \frac{1}{2} \cdot \frac{2012}{2011}
\]
4. Find the limit \( \lim_{n \to \infty} (n!)^{\frac{1}{n-1}}. \)

**Solution:** We have that \( n! > \left( \frac{n-1}{2} \right)^{n-1} \) and this gives \( (n!)^{\frac{1}{n-1}} > \sqrt[2]{n-1} \). Then \( \lim_{n \to \infty} (n!)^{\frac{1}{n-1}} = \infty. \)

Answer: \( \infty \)

5. Given the numbers \( e^\pi, 2^3, \pi^e, 3^2, 3^e \) put them in increasing order.

**Solution:** It is clear that \( 2^3 < 3^2 < 3^e < \pi^e \). Then we need to show \( \pi^e < e^\pi \). This is equivalent to showing \( \frac{\ln \pi}{\pi} < \frac{\ln e}{e} = \frac{1}{e} \). Let \( \phi(x) = \frac{\ln x}{x} \). It is easy to see that \( \phi'(x) < 0 \) for all \( x > e \). Therefore the function \( \phi(x) \) is decreasing on the interval \([e, \pi]\) and we conclude \( \frac{\ln e}{e} = \phi(e) > \phi(\pi) = \frac{\ln \pi}{\pi} \).

Answer: \( 2^3, 3^2, 3^e, \pi^e, e^\pi \)

6. Evaluate \( \lim_{n \to \infty} \int_0^1 \frac{ny^{n-1}}{2011+y} \, dy \)

**Solution:** Integration by parts with \( u = \frac{1}{2011+y} \) and \( dv = ny^{n-1} \, dy \) gives

\[
\int_0^1 \frac{ny^{n-1}}{2011+y} \, dy = \left. \frac{y^n}{2011+y} \right|_0^1 + \int_0^1 \frac{y^n}{(2011+y)^2} \, dy.
\]

Since

\[
0 \leq \int_0^1 \frac{y^n}{(2011+y)^2} \, dy \leq \frac{1}{2011^2} \int_0^1 y^n \, dy \leq \frac{1}{2011^2(n+1)}
\]

we conclude that the limit of the above term is zero which gives the desired limit

\[
\lim_{n \to \infty} \int_0^1 \frac{ny^{n-1}}{2011+y} \, dy = \lim_{n \to \infty} \frac{y^n}{2011+y} \bigg|_0^1 = \frac{1}{2012}
\]

Answer: \( \frac{1}{2012} \)

7. A rectangle is inscribed in a sector of a circle of radius 1 and angle \( \theta \) as shown in Figure 1. Find the maximum area of such rectangle if \( \theta = \frac{\pi}{3} \).

**Solution:** Let \( \alpha \) be the angle shown in the Figure 1. We will express the area \( A = AB \cdot BC \) of the rectangle \( ABCD \) as a function of \( \alpha \). Since \( \tan \theta = \frac{AD}{OA} \) and \( BC = OC \sin \alpha = \sin \alpha \), we have

\[
AB = OB - OA = OC \cos \alpha - AD \cot \theta = \cos \alpha - \sin \alpha \cot \theta
\]
Then, we obtain that the area of the rectangle is
\[ A(\alpha) = AB \cdot BC = \frac{1}{2} \sin 2\alpha - \sin^2 \alpha \cot \theta. \]

The area is maximal if \( A'(\alpha) = 0 \), recall that \( 0 \leq \alpha \leq \theta = \pi/3 \), \( A(0) = A(\theta) = 0 \). Therefore \( \alpha = \theta/2 \) and the maximal area is
\[ A(\theta/2) = \frac{1}{2} \sin \theta - \sin^2(\theta/2) \cot \theta = \frac{1 - \cos \theta}{2 \sin \theta} \]

If the angle is \( \theta = \pi/3 \) then the maximal area is \( A = \frac{1/2}{\sqrt{3}} \). Note that it is easy to guess that the maximal area is at \( \alpha = \theta/2 \) because the area as a function of \( \alpha \) is increasing at \( \alpha = 0 \) and decreasing at \( \alpha = \theta \).

Answer: \( \frac{1}{2\sqrt{3}} \)

8. You have one fair coin and one bias coin with \( \text{Prob(heads)} = 2/3 \). One of the coins is tossed once, resulting in heads. The other one is tossed three times, resulting in two heads and one tails. There are two possible ways to have that outcome. Find the probability of the more likely event.

Solution: If the fair coin is the one flipped once and the other one is flipped three times we have
\[ P_1 = \frac{1}{2} \left( \frac{3}{2} \right) \left( \frac{2}{3} \right)^2 \left( \frac{1}{3} \right)^1 = \frac{2}{9}. \]

If the biased coin is the one flipped once then
\[ P_2 = \frac{2}{3} \left( \frac{3}{2} \right) \left( \frac{1}{2} \right)^2 \left( \frac{1}{2} \right)^1 = \frac{1}{4} > P_1. \]
Hence, the more likely event has probability $\frac{1}{4}$.

Answer: $\frac{1}{4}$

9. Find the area of the portion of the $xy$-plane enclosed by the curve with equation

$$|x - 0.5| + |x + 0.5| + \frac{2|y|}{\sqrt{3}} = 2.$$ 

Solution: The equation is symmetric with respect to changing $x$ to $-x$ and $y$ to $-y$, and the enclosed portion is between the lines $x = 1$ and $x = -1$. If $-1 \leq x \leq -1/2$ then $y = \pm \sqrt{3}(x + 1)$. If $-1/2 \leq x \leq 1/2$ then $y = \pm \sqrt{3}/2$. If $1/2 \leq x \leq 1$ then $y = \pm \sqrt{3}(1 - x)$. Therefore the curve is a regular hexagon with vertices $(\pm 1, 0)$, $(\pm 1/2, \sqrt{3}/2)$, and $(\pm 1/2, -\sqrt{3}/2)$. The side of the hexagon is 1 and this gives the area $A = 6 \cdot \frac{\sqrt{3}}{4}$.

Answer: $\frac{3\sqrt{3}}{2}$

10. A fair coin is tossed 10 times. Find the probability that two tails do not appear in succession.

Solution: There are $2^{10} = 1024$ possible outcomes if the coin is tossed 10 times. Let $H_n$ be the number of $n$-toss sequences which do not have two or more tails in succession but ending with heads for the last toss. Let $T_n$ be the number of $n$-toss sequences which do not have two or more tails in succession but ending with tails for the last toss. Then $H_2 = 2$, $T_2 = 1$ and $H_1 = 1$, and for any $n \geq 3$, we have $T_n = H_{n-1}$ and $H_n = H_{n-1} + T_{n-1}$. Therefore, $H_n = H_{n-1} + H_{n-2}$, and we compute

$H_3 = 3, H_4 = 5, H_5 = 8, H_6 = 13, H_7 = 21, H_8 = 34, H_9 = 55, H_{10} = 89$.

Then the probability is $\frac{89}{1024}$

Answer: $\frac{89}{1024}$

11. The quadratic equation $ax^2 + bx + c = 0$ has integer coefficients $a, b, c$. Which of the following numbers cannot be the discriminant of the equation?

$4, 9, 16, 17, 21, 23, 24, 25, 28, 33, 36$

Solution: It is easy to verify that all numbers but 23 could be of the type $D = b^2 - 4ac$. If we assume that $23 = b^2 - 4ac$ for some integers $a, b, c$, then $b$ should be odd: $b = 2k + 1$ for some integer $k$. Then we have $23 = 4k^2 + 4k + 1 - 4ac$ which gives $22 = 4(k^2 + k - ac)$ and this is impossible.
Answer: 23

12. If \([x]\) is the greatest integer less than or equal to \(x\), then compute the following sum

\[
S = \sum_{N=1}^{2011} [\log_2 N]
\]

**Solution:** We have the following values for \([\log_2 N]\): \([\log_2 N] = k\) for \(2^k \leq N < 2^{k+1}\), for any \(k = 1, 2, \ldots\). Then the sum is

\[
S = 1(2^2 - 2) + 2(2^3 - 2^2) + \cdots + 9(2^{10} - 2^9) + 10(2011 - 1023)
\]

We rearrange the terms

\[
S = 9 \cdot 2^{10} - (2^9 + 2^8 + \cdots + 2^2 + 2) + 9880 = 8 \cdot 2^{10} + 2 + 9880 = 18074.
\]

Answer: 18074

13. Compute the angle \(\theta = \tan^{-1}\left(\frac{\sin(\pi/18) + \sin(2\pi/18)}{\cos(\pi/18) + \cos(2\pi/18)}\right)\). Express your answer in degrees.

**Solution:** We have that

\[
\frac{\sin(\pi/18) + \sin(2\pi/18)}{\cos(\pi/18) + \cos(2\pi/18)} = \frac{2\sin(3\pi/36)\cos(\pi/36)}{2\cos(3\pi/36)\cos(\pi/36)} = \tan(3\pi/36)
\]

The angle \(3\pi/36\) radians is equal to 15 degrees.

Answer: 15

14. How many integers between 1 and 2011 have exactly 27 positive divisors?

**Solution:** Suppose that \(n\) has exactly 27 divisors. Note that if the factorization of \(n\) is \(p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}\) with \(p_1, p_2, \ldots, p_k\) distinct primes and all exponents \(m_i \geq 1\), then the number of divisors of \(n\) is \((m_1 + 1)(m_2 + 1) \cdots (m_k + 1) = 27\). The only possible options for \(k\) are \(k = 1, 2, 3\). If \(k = 1\) the smallest number of this type is \(2^{26} > 2011\). If \(k = 2\) the smallest number of this type is \(2^83^2 > 2011\). The only numbers of this type between 1 and 2011 occur when \(k = 3\) and \(m_1 = m_2 = m_3 = 2\). Checking some possibilities we have

\[
2^25^27^2 = 4900 \text{ (too big)} \quad 2^23^27^2 = 1764 \text{ (just right)}
\]

Thus, there are two integers with this property: \(2^23^27^2\) and \(2^23^25^2\).

Answer: 2
15. Find the number of real solutions to the system

\[ 2y = x + \frac{2011}{x}, \quad 2z = y + \frac{2011}{y}, \quad 2w = z + \frac{2011}{z}, \quad 2x = w + \frac{2011}{w}. \]

**Solution:** For any positive number \( a \), we have

\[ \frac{1}{2} \left( a + \frac{2011}{a} \right) \geq \sqrt{2011} \]

because this inequality is equivalent to \((a - \sqrt{2011})^2 \geq 0\). If \( x > 0 \) we derive

\[ y = \frac{1}{2} \left( x + \frac{2011}{x} \right) \geq \sqrt{2011} \]

and using now that \( y > 0 \) and so on, we consecutively derive \( z \geq \sqrt{2011}, w \geq \sqrt{2011}, \) and \( x \geq \sqrt{2011} \). Then we have

\[ x - \frac{2011}{x} \geq 0, \quad y - \frac{2011}{y} \geq 0, \quad z - \frac{2011}{z} \geq 0, \quad w - \frac{2011}{w} \geq 0. \]

Adding the four equations of the original system we obtain

\[ (x - \frac{2011}{x}) + (y - \frac{2011}{y}) + (z - \frac{2011}{z}) + (w - \frac{2011}{w}) = 0. \]

Each term in parentheses above is nonnegative and we conclude the only solution for \( x > 0 \) is \( x = y = z = w = \sqrt{2011} \). By symmetry, each positive solution corresponds to a negative solution: \( x = y = z = w = -\sqrt{2011} \). Therefore, there are two solutions of the system.

Answer: 2

16. Find the greatest integer less than \[ 10 \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \cdots}}}}. \]

**Solution:** Let

\[ S_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \cdots + \sqrt{n}}}}}. \]

It is easy to show that \( S_{n+1} > S_n \) and for \( x \geq 6 \) we have the inequality

\[ x + \sqrt{2} x < 2x - 2 \]
We have
\[ S_n < \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n - 1}}} + \sqrt{2(n - 1)}}, \]
and using the inequality with \( x = n - 1 \) above we get
\[ S_n < \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n - 2}}} + \sqrt{2(n - 2)}}. \]

We apply the same argument for \( x = n - 2, n - 3, \ldots, 5 \) and obtain
\[ S_n < \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \sqrt{5 + \sqrt{10}}}}} < \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4}}} = \sqrt{1 + \sqrt{5}}}. \]

This proves that the sequence \( S_n \) is increasing and bounded above. Hence, it has a limit \( S \). We have the following bounds
\[ 10\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4}}} < S < 10\sqrt{1 + \sqrt{5}}}. \]

It is easy to check that \( 10\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4}}} > 17 \) and \( 10\sqrt{1 + \sqrt{5}} < 18 \) which gives that the greatest integer less than \( S \) is 17.

Answer: 17

17. Find the ratio of the shaded area and the total area of the triangle as shown in Figure 2 if each side of the big triangle is divided in two pieces with 2 : 1 ratio.

Solution: Let’s denote the areas of the small triangles with \( a, b, c, d, e, f, g, h, i, j \) as shown in Figure 2. Using the 2:1 ratio of the sides, it is easy to derive
\[ 2a = d, \quad 2(a + h) = d + g + j, \]
\[ 2(a + h + e + b) = d + g + j + c + f + i. \]

We can simplify these three equations to
\[ 2a = d, \quad 2h = g + j, \quad 2(e + b) = c + f + i. \]

Similarly, we derive the same type of relationships starting from \( b \):
\[ 2b = e, \quad 2i = h + j, \quad 2(f + c) = a + d + g. \]
and one more time starting from $c$:

$$2c = f, \quad 2g = i + j, \quad 2(d + a) = b + e + h.$$ 

Using these nine equations (we add them in groups of three) we derive

$$2(a + b + c) = d + e + f, \quad h + i + g = 3j, \quad a + b + c + e + f + d = h + i + g.$$ 

This implies that the shaded area $j$ is

$$j = \frac{1}{7}(a + b + c + d + e + f + g + h + i + j).$$ 

Therefore, the ratio of the shaded area and the total area is $\frac{1}{7}$.

**Answer:** $\frac{1}{7}$

18. Let $D$ be a 9-digit number of the form $d_1d_2\cdots d_9$ with $d_i \neq 0$ for all $i$. Form a 9-digit number $E = e_1e_2\cdots e_9$ such that $e_i \neq 0$ is a digit which causes $D$ to be divisible by 7 when $d_i$ is replaced by $e_i$ (for instance, if $D = 123456789$, $e_2 = 4$ since 143456789 is divisible by 7). Similarly, form a 9-digit number $F = f_1f_2\cdots f_9$ such that $f_i \neq 0$ is a digit which causes $E$ to be divisible by 7 when $e_i$ is replaced by $f_i$. If $D = 123456789$, find the product of all possible values of $f_1$.

**Solution:** For $i = 1, \ldots, 9$, $(e_i - d_i) \times 10^{9-i} + D$ is divisible by 7. Adding these together yields $E + 8D$ which is also divisible by 7, meaning $E + D$ is also divisible by 7. Similarly, $(f_1 - e_1) \times 10^8 + E$ and $(e_1 - d_1) \times 10^8 + D$ are divisible by 7, so their sum, $(f_1 - d_1) \times 10^8 + E + D$ is divisible by 7. Therefore, $f_1 - d_1$ must be divisible by 7, so $f_1 = 1$ or $f_1 = 8$. Each of these values can appear in $F$. For example, $E = 449137331$ is a number of the desired form, and $F = 123456789$ or $F = 823456789$ are the possible for this $E$. Therefore, the product of all possible values of $f_1$ is 8.
19. Jenny and Kenny are walking in the same direction, Kenny at 3 feet per second and Jenny at 1 foot per second, on parallel paths that are 200 feet apart. A tall circular building 100 feet in diameter is centered midway between the paths. At the instant when the building first blocks the line of sight between Jenny and Kenny, they are 200 feet apart. Find $t$, time in seconds, before Jenny and Kenny can see each other again.

Solution: Assume that the circle of radius 50 (base of the building) is centered at $(0,0)$. Assume also that they start at points $(-50,100)$ and $(-50,-100)$ (200 ft apart). Then at time $t$, they end up at points $(-50+t,100)$ and $(-50+3t,-100)$. The equation of the line connecting these points and the equation of the circle are $y = -\frac{100}{t}x + 200 - \frac{5000}{t}$ and $x^2 + y^2 = 50^2$. When they see each other again, the line connecting the two points will be tangent to the circle at the point $(x,y)$. Since the radius is perpendicular to the tangent we get $-\frac{x}{y} = -\frac{100}{t}$ or $xt = 100y$. Substitute $y = \frac{xt}{100}$ into $x^2 + y^2 = 50^2$ and get $x = \frac{5000}{\sqrt{100^2 + t^2}}$. Now substitute this and $y = \frac{xt}{100}$ into $y = -\frac{100}{t}x + 200 - \frac{5000}{t}$ and solve for $t$ to get $t = \frac{160}{3}$.

Answer: $\frac{160}{3}$ sec