1. One cow gives \( \frac{x+1}{x+3} \) cans of milk in \( x + 2 \) days, hence one cow gives \( \frac{x+1}{2(x+3)} \) cans of milk per day. It follows that it will take \( \frac{x+5}{x+3} \cdot \frac{x(x+2)}{x+1} \) days for \( x + 3 \) cows to give \( x + 5 \) cans of milk.

2. We have \( \frac{x+2}{x+3} = 2\sqrt{ab} \), hence \( a + b = 4\sqrt{ab} \), or \( ab + 1 = 4\sqrt{ab} \); and we get a quadratic equation \( x^2 - 4x + 1 = 0 \) for \( \sqrt{ab} \). It has solutions \( x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3} \). We have \( 2 < \sqrt{3} < 1 \), since \( 1 < \sqrt{3} \). But \( \sqrt{a/b} > 1 \), hence the only solution is \( \sqrt{a/b} = 2 \pm \sqrt{3} \), or \( x/b = (2 \pm \sqrt{3})^2 = 7 \pm 4\sqrt{3} \).

3. Let \( a \) and \( b \) be velocities of \( A \) and \( B \), respectively (signed, and measured in yards per minute). Then \( |2a| = |500 + 2b| \), and \( |10a| = |500 + 10b| \). Dividing by \( |a| \), we get \( 2 = |(500 + 2b)/a| \) and \( 10 = |(500 + 10b)/a| \), hence \( |(2500 + 10b)/a| = |(500 + 10b)/a| \).

Then either \( (2500 + 10b)/a = (500 + 10b)/a \), or \( (2500 + 10b)/a = -(500 + 10b)/a \).

Multiplying by \( a \), we get that either

\[
2500 + 10b = 500 + 10b, \text{ or } 2500 + 10b = -500 - 10b.
\]

The first case is impossible, hence 3000 = -20b, so that \( b = -150 \). From \( |2a| = |500 + 2b| \) we get that \( |2a| = 200 \), and \( |a| = 100 \). The answers are \( |a| = 100 \) and \( |b| = 150 \).

4. The diagonals of the rhombus divide it into four congruent right triangles of area \( 5 \cdot 12/2 = 30 \). Their hypotenuse (the side of the rhombus) has length \( \sqrt{25 + 144} = \sqrt{169} = 13 \), hence the height to the hypotenuse of the triangle has length \( 60/13 \). Since the lengths of these heights are the same, they are equal to the radius of the inscribed circle.

5. Consider the five cases \( x \leq -1 \), \( -1 < x \leq 0 \), \( 0 < x \leq 1 \), \( 1 < x \leq 2 \), and \( x > 2 \). In the first case the equation is equivalent to

\[-(x + 1) + x - 3(x - 1) + 2(x - 2) = x + 2, \quad -x - 2 = x + 2, \quad x = -2;\]

in the second case to:

\[(x + 1) - x - 3(x - 1) + 2(x - 2) = x + 2, \quad x = x + 2, \quad x = -1,\]

which is a contradiction; in the third case to

\[(x + 1) - x - 3(x - 1) + 2(x - 2) = x + 2, \quad -x = x + 2, \quad x = -1,\]

which does not satisfy the inequalities defining the case; in the fourth case to

\[(x + 1) - x + 3(x - 1) + 2(x - 2) = x + 2, \quad 5x - 6 = x + 2, \quad x = 2;\]

in the fifth case to

\[(x + 1) - x + 3(x - 1) - 2(x - 2) = x + 2, \quad x + 2 = x + 2, \quad x = 2;\]

which is satisfied for all \( x \) satisfying the inequalities of the case.

Consequently, the set of solutions is \( \{-2\} \cup \{x \geq 2\} \).

6. Connecting the center of the circle with the vertices of the triangle, we see that the area \( K \) of the triangle is equal to \( Pr/2 \). It follows that \( P/K = 2/r \).

7. Exponentiating, we get 5\(^a\)2\(^b\) = 200\(^c\) = 8\(^d\)25\(^e\) = 2\(^3\)5\(^2\)c. Then \( a = 2c \) and \( b = 3c \). Since \( a, b, c \) have no common factor greater than 1, \( c \) is equal to 1, as it is a positive common factor. It follows that \( a = 2 \) and \( b = 3 \).
8. The set of the points of the form \((5 \cos \theta, 5 \sin \theta)\) is the circle of radius 5 with center in \((0, 0)\). The set of vertices \((x, y)\) such that the triangle with vertices in \((-5, 0), (5, 0), (x, y)\) has area 10 is the union of the lines \(y = 2\) and \(y = -2\), since this is the set of points such that the corresponding triangle has height of length 2 to the side connecting \((-5, 0)\) and \((5, 0)\). It is clear that these two lines intersect the circle in four points.

9. The remainder is equal to the value of the polynomial at \(x = -2\), which is \(16 - 16 - 2k + 3 = -2k + 3\). It is equal to \(k\) if and only if \(k = 1\).

10. The first equation is equivalent to \((x - 6)^2 + (y - 3)^2 = 4 + 36 + 9 = 49\). The second equation is equivalent to \((x - 2)^2 + (y - 6)^2 = k + 4 + 36 = k + 40\). It follows that the first curve is the circle with center \((6, 3)\) and radius 7, while the second curve is the circle with center \((2, 6)\) and radius \(\sqrt{k + 40}\). The distance between the centers is \(\sqrt{16 + 9} = 5\), hence they intersect if and only if the radius 7 - 5 \(\leq \sqrt{k + 40} \leq 7 + 5\), which is equivalent to \(4 \leq k + 40 \leq 144\), or to \(-36 \leq k \leq 104\).

11. If we multiply all three equalities, we will get \(xyz + x + y + z + 1/x + 1/y + 1/z + 1/xyz = 28/3\). Subtracting from this equality the sum of the first three equalities, we get \(xyz + 1/xyz = 28/3 - 22/3 = 2\). Multiplying this by \(xyz\), we get \((xyz)^2 - 2xyz + 1 = 0\), or \((xyz - 1)^2 = 0\), which implies that \(xyz = 1\).

12.

Let \(O, P, C\) be the centers of the circles \(L, M, K\), respectively. Let \(D\) be the tangency point of \(AB\) with \(M\), and let \(PQ\) be perpendicular to \(OC\). Denote by \(R\) and \(r\) the radii of \(L\) and \(M\), respectively. Then \(2R\) is the radius of the circle \(K\). The centers of tangent circles and the point of tangency are collinear. Consequently, continuation of \(CP\) intersects the circle \(K\) at the tangency point of \(K\) with \(M\), and so \(CP = 2R - r\). By Pythagoras’ Theorem, \(CD = \sqrt{(2R - r)^2 - r^2} = \sqrt{4R^2 - 4Rr}\). We have \(QC = PD = r\), \(QP = CD = \sqrt{4R^2 - 4Rr}\), \(OP = R + r\), and \(OQ = OC - CQ = R - r\). Applying Pythagoras’ Theorem to \(\triangle OPQ\), we get \((R + r)^2 = (R - r)^2 + 4R^2 - 4Rr\), or

\[
R^2 + 2Rr + r^2 = R^2 - 2Rr + r^2 + 4R^2 - 4Rr, \quad 8Rr = 4R^2, \quad 2r = R.
\]

The area of the circle \(K\) is then \(\pi (2R)^2 = 4\pi R^2 = 16\pi r^2\). The area of \(M\) is \(\pi r^2\). Hence, the ratio is 16.

13.

Let us assume that the length of the side of the square is 1 unit. Then the area of one of the smaller triangles is \(m\) square units, hence the lengths of its legs are 1 and 2\(m\) units. Suppose that the lengths of the legs of the other small triangle are 1 and \(x\) units. These triangles are similar, hence we get the proportion

\[x : 1 = 1 : 2m,\]
that implies \( x = \frac{1}{2\pi} \). Hence, the area of the other small triangle is \( \frac{1}{4\sin^2 \frac{\theta}{2}} \) square units. It follows that the ratio of the area of the other small triangle to the area of the square is \( \frac{1}{4\sin^2 \frac{\theta}{2}} \).

14. We get \( \sin^2 \frac{\theta}{2} = \frac{x^2 - 1}{2x} \), hence \( \cos^2 \frac{\theta}{2} = \frac{x^2 + 1}{2x} \). Using the double angle formulas, we get

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \frac{2\sqrt{\frac{x^2 - 1}{2x}} \sqrt{\frac{x^2 + 1}{2x}}}{\frac{x^2 + 1}{2x} - \frac{x^2 - 1}{2x}} = \sqrt{x^2 - 1}.
\]

15. We have \( N = 49A + 7B + C = 81C + 9B + A \). Subtracting one expression from the other, we get \( 48A - 2B - 80C = 0 \), or \( 24A - 40C = B \). It follows that \( B \) is divisible by 8. But it is a digit for base 7 system, so it can not be greater than 6. Consequently, \( B = 0 \), and \( 24A = 40C \), or \( 3A = 5C \). It follows that \( A \) is divisible by 5, hence it is equal to 5 (since it can not be zero, as it is the first digit of a number, and it can not be bigger than 6). Then \( C = 3 \), and \( N = 49 \cdot 5 + 3 = 81 \cdot 3 + 5 = 248 \).

16. We have \( y^5 = x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} - x^{-5} \), \( y^3 = x^3 - 3x + 3x^{-1} - x^{-3} \), hence \( y^5 + 5y^3 = x^5 - 5x + 5x^{-1} - x^{-5} \), hence \( x^5 - x^{-5} = y^5 + 5y^3 + 5y \).

17. If we divide it by 2, then it becomes equal to \( \sin \alpha \cos 60^\circ - \cos \alpha \sin 60^\circ = \sin(\alpha - 60^\circ) \). Which is minimal when \( \sin(\alpha - 60^\circ) = -1 \), i.e., when \( \alpha = 330^\circ \).

18. When we open brackets, \( x^7 \) is obtained as a product \( x^{k_1}x^{k_2}x^{k_3}x^{k_4} \), where \( k_i \) is either zero, or one, or two and such that \( k_1 + k_2 + k_3 + k_4 = 7 \). If one of \( k_i \) is zero, then the sum of the other three is at most 6. It follows that the only possibilities for \( (k_1, k_2, k_3, k_4) \) are:

\[
(1, 2, 2, 2), (2, 1, 2, 2), (2, 2, 1, 2), (2, 2, 2, 1)
\]

The coefficient at a monomial \( x^{k_1}x^{k_2}x^{k_3}x^{k_4} \) where \( (k_1, k_2, k_3, k_4) \) is one of the above, is equal to \( -2 \). Consequently, the coefficient at \( x^7 \) is \( -8 \).

19. Let us multiply the original equation by \( \sec x + \tan x \). We get \( \sec^2 x - \tan^2 x = 2(\sec x + \tan x) \), but \( \sec^2 x - \tan^2 x = \frac{1 - \sin^2 x}{\cos^2 x} = 1 \). Consequently, \( \sec x + \tan x = 1/2 \).

20. Denote \( x = a^{c/4}, y = b^{c/4} \). Then \( a^c + b^c - (ab)^{c/4} = x^4 + y^4 - xy \) and we have

\[
x^4 + y^4 - xy = (x^2 - y^2)^2 + 2x^2y^2 - xy = (x^2 - y^2)^2 + 2 \left( x^2y^2 - \frac{1}{2}xy + \frac{1}{16} \right) - \frac{1}{8} = (x^2 - y^2)^2 + 2 \left( x^2y^2 - \frac{1}{4} \right)^2 - \frac{1}{8} \geq -\frac{1}{8}.
\]

The equality takes place if \( x^2 = y^2 \) and \( x^2y^2 = 1/4 \), i.e., if \( x^2 = y^2 = 1/2 \). Consequently, the minimum value is \(-1/8\).

21. Suppose that \( x^2 - x + a \) divides \( x^{13} + x + 90 \). Then, substituting \( x = 0 \) and \( x = 1 \) we get that \( a \) divides \( 90 \), and \( a \) divides \( 92 \). Consequently, \( a \) divides g.c.d. of \( 90 \) and \( 92 \), which is equal to 2. It follows that \( a \in \{1, -1, 2, -2\} \).

Let us substitute \( x = -1 \). We get that \( a + 2 \) divides \( 88 \). This rules out \( a = 1 \) and \( a = -2 \), so that we have \( a \in \{-1, 2\} \).

Dividing \( x^{13} + x + 90 \) by \( x^2 - x - 1 \) we get quotient \( x^{11} + x^{10} + 2x^9 + 3x^8 + 5x^7 + 8x^6 + 13x^5 + 21x^4 + 34x^3 + 55x^2 + 89x + 144 \) (note that beginning coefficients form the Fibonacci sequence) and remainder \( 234x + 144 \), hence the only candidate is \( a = 2 \). In fact, one can check directly that \( x^{13} + x + 90 \) is divisible by \( x^2 - x + 2 \) (though it is not required according to formulation of the problem).

22. It follows that \( P(1) = P(0^2 + 1) = 0 + 1 = 1 \), \( P(2) = P(1^2 + 1) = P(1)^2 + 1 = 2 \), \( P(5) = P(2^2 + 1) = 2^2 + 1 \), etc. We will get an infinite sequence \( a_n \) given recursively by \( a_{n+1} = a_n^2 + 1 \), and a proof by induction that \( P(a_n) = a_n \):

\[
P(a_{n+1}) = P(a_n^2 + 1) = (P(a_n))^2 + 1 = a_n^2 + 1 = a_{n+1}.
\]
We get hence two polynomials: $P(x)$, and $Q(x) = x$, such that $P(x) - Q(x) = 0$ for infinitely many values of $x$. But this is possible only when $P(x)$ and $Q(x)$ are identically equal, since a polynomial of degree $n$ has not more than $n$ zeros.